Advanced Data Structures

String Pattern Matching/Text Search

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What is Pattern Matching?

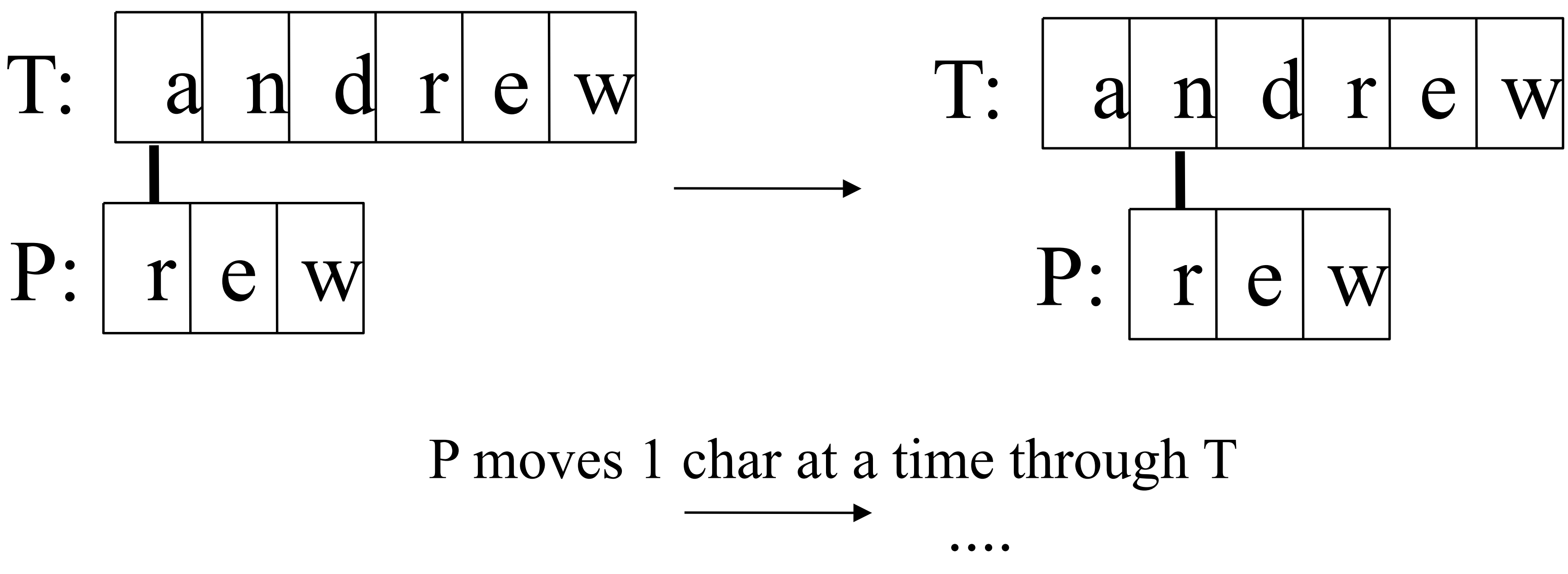
- Definition:
  - given a text string T and a pattern string P, find the pattern inside the text
    - T: “the rain in spain stays mainly on the plain”
    - P: “n th”

Text search

- Pattern matching directly
  - Brute force
  - KMP
  - BM
- Regular expressions (Not in this course)
- Indices for pattern matching
  - Inverted files
  - **Signature files**
  - **Suffix trees** and **Suffix arrays**

The Brute Force Algorithm

- Check each position in the text T to see if the pattern P starts in that position



Analysis

- Brute force pattern matching runs in time O(mn) in the worst case.
- But most searches of ordinary text take O(m+n), which is very quick.

continued

- The brute force algorithm is fast when the alphabet of the text is large
  - e.g. A..Z, a..z, 1..9, etc.

- It is slower when the alphabet is small
  - e.g. 0, 1 (as in binary files, image files, etc.)

continued

- Example of a worst case:
  - T: "aaaaaaaaaaaaaaaaaaaaaaaaaah"
  - P: "aaah"

- Example of a more average case:
  - T: "a string searching example is standard"
  - P: "store"

The KMP Algorithm

- The Knuth-Morris-Pratt (KMP) algorithm looks for the pattern in the text in a *left-to-right* order (like the brute force algorithm).

- But it shifts the pattern more intelligently than the brute force algorithm.

continued

Summary

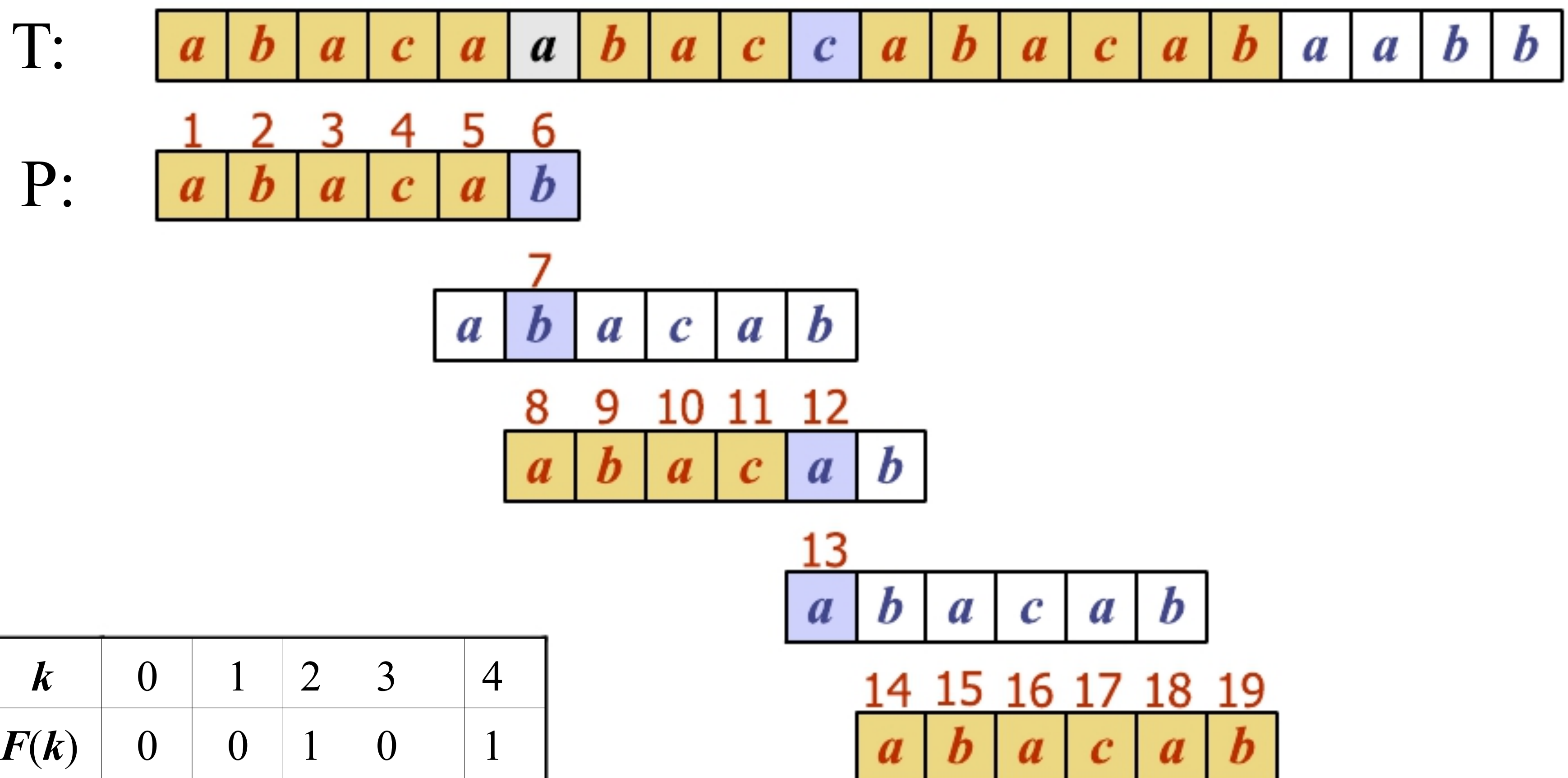
- If a mismatch occurs between the text and pattern P at P[j], what is the *most* we can shift the pattern to avoid wasteful comparisons?

Summary

- If a mismatch occurs between the text and pattern P at P[j], what is the *most* we can shift the pattern to avoid wasteful comparisons?

- *Answer*: the largest prefix of P[0 .. j-1] that is a suffix of P[1 .. j-1]

Example



k	0	1	2	3	4
***F*(*k*)**	0	0	1	0	1

KMP Advantages

- KMP runs in optimal time: O(m+n)
  - very fast
- The algorithm never needs to move backwards in the input text, T
  - this makes the algorithm good for processing very large files that are read in from external devices or through a network stream

KMP Disadvantages

- KMP doesn’t work so well as the size of the alphabet increases
  - more chance of a mismatch (more possible mismatches)
  - mismatches tend to occur early in the pattern, but KMP is faster when the mismatches occur later

Boyer and Moore Algorithm

A fast string searching algorithm. *Communications of the ACM*. Vol. 20 p.p. 762-772, 1977.

BOYER, R.S. and **MOORE, J.S.**

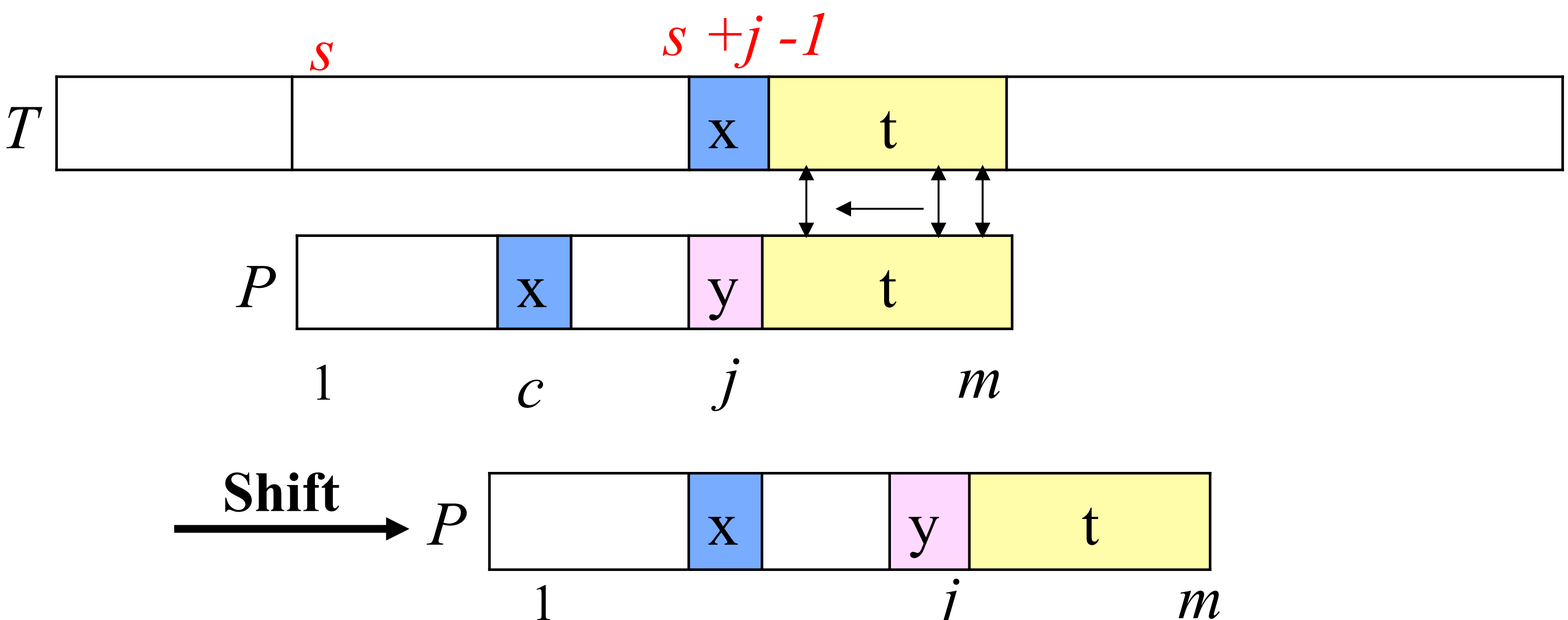
Boyer and Moore Algorithm

- The algorithm compares the pattern *P* with the substring of sequence *T* within a sliding window in the **right-to-left order**.
- The **bad character rule** and **good suffix rule** are used to determine the movement of sliding window.

Bad Character Rule

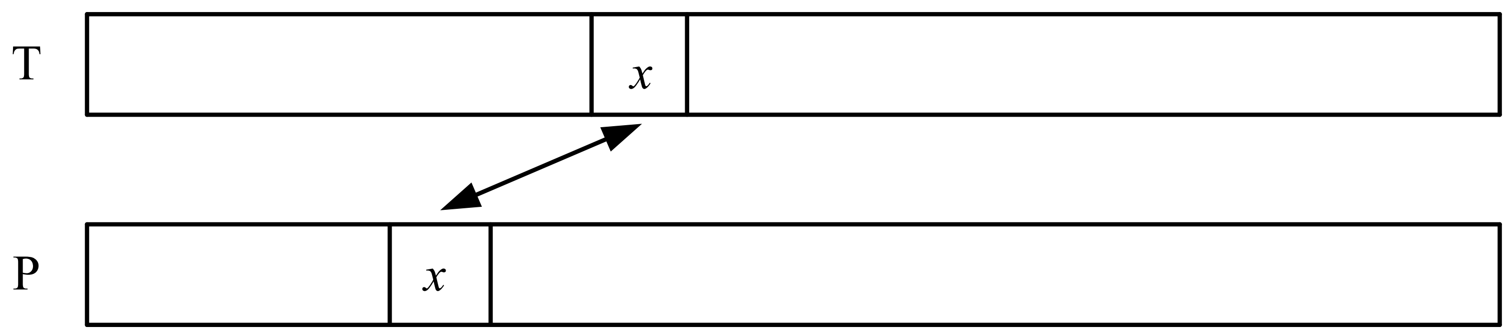
Suppose that P_1 is aligned to T_s now, and we perform a pair-wise comparing between text T and pattern P from right to left. Assume that the first mismatch occurs when comparing T_{s+j-1} with P_j .

Since $T_{s+j-1} \neq P_j$, we move the pattern P to the right such that the largest position c in the left of P_j is equal to T_{s+j-1}. We can shift the pattern at least (j-c) positions right.



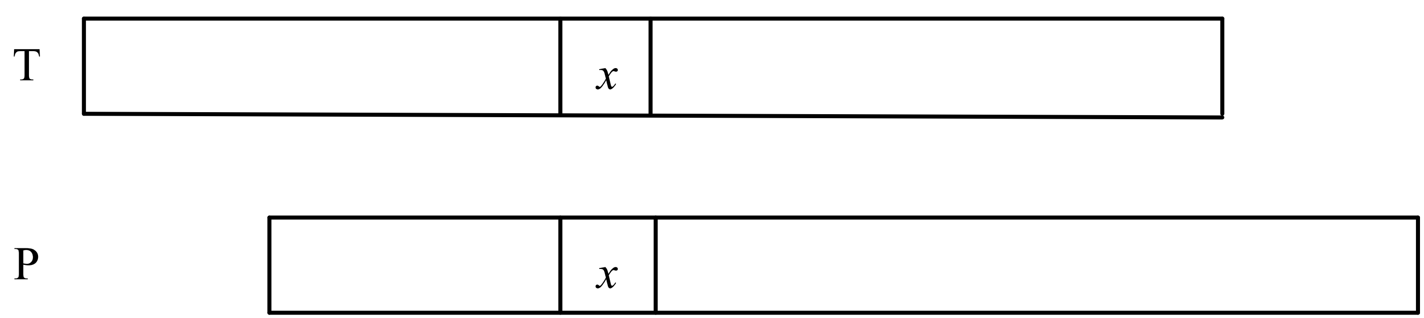
Character Matching Rule

- Bad character rule uses Rule 2-1 (Character Matching Rule).
- For any character x in T, find the nearest x in P which is to the left of x in T.

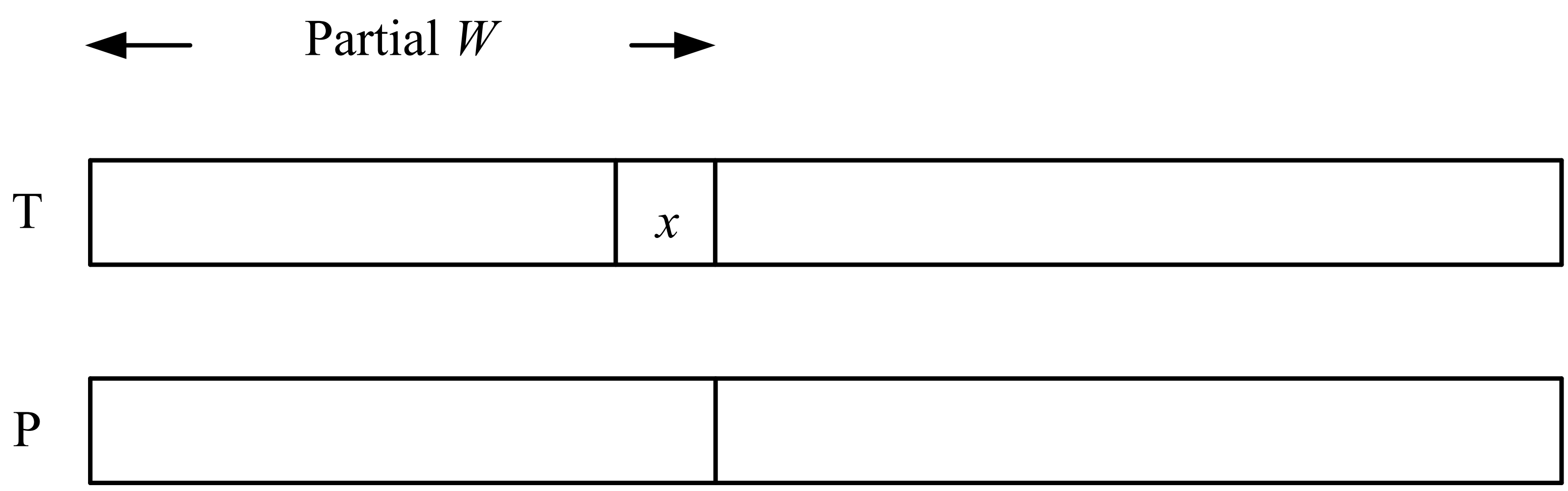


Implication

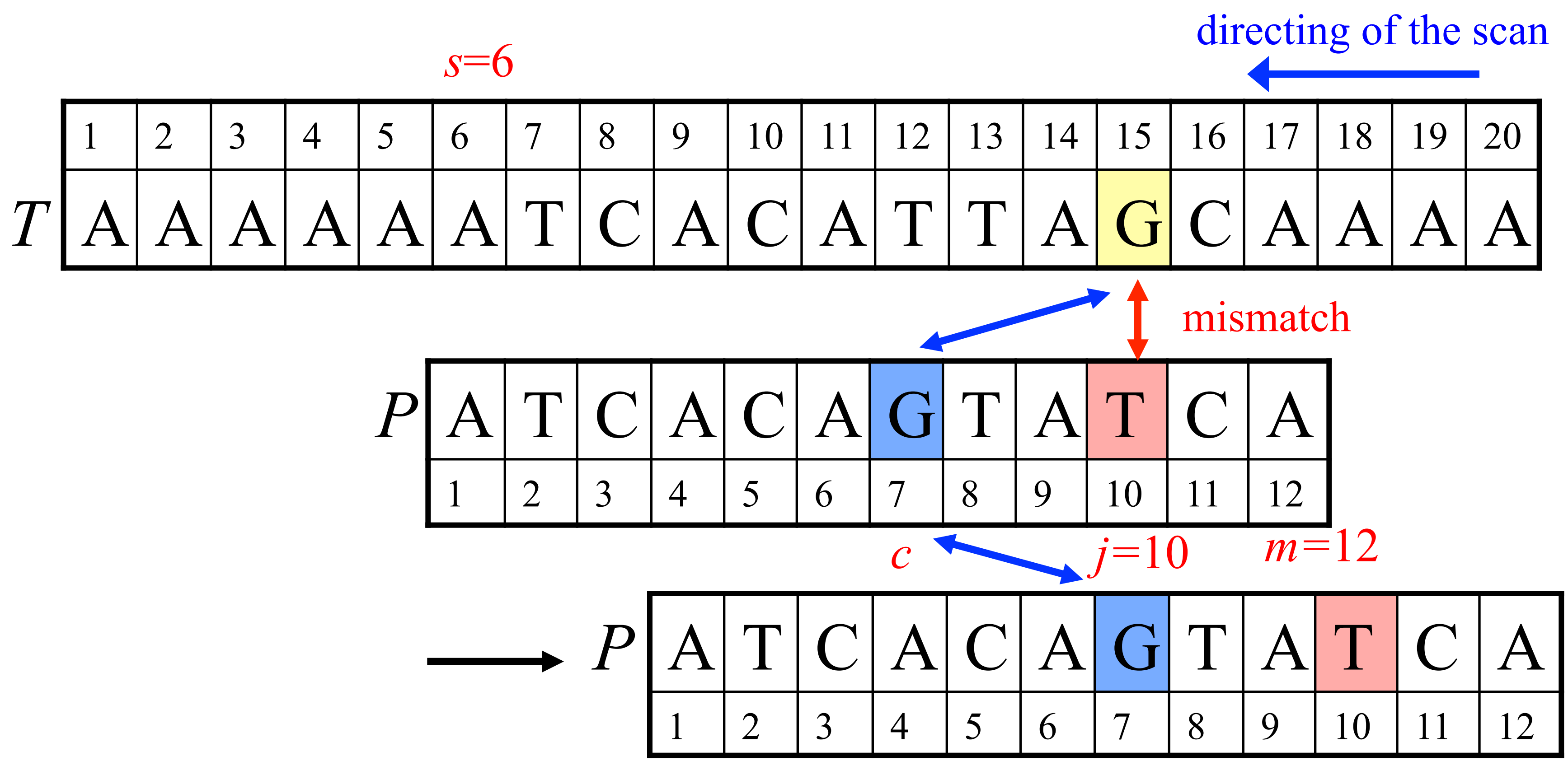
- Case 1. If there is a *x* in *P* to the left of *T*, move *P* so that the two x's match.



- Case 2: If no such a x exists in P, consider the partial window defined by x in T and the string to the left of it.



- Ex: Suppose that P_1 is aligned to T_6 now. We compare pair-wise between T and P from right to left. Since $T_{16,17} = P_{11,12} =$ "CA" and $T_{15} =$ "G" $\neq P_{10} =$ "T". Therefore, we find the rightmost position $c=7$ in the left of P_{10} in P such that P_c is equal to "G" and we can move the window at least (10-7=3) positions.



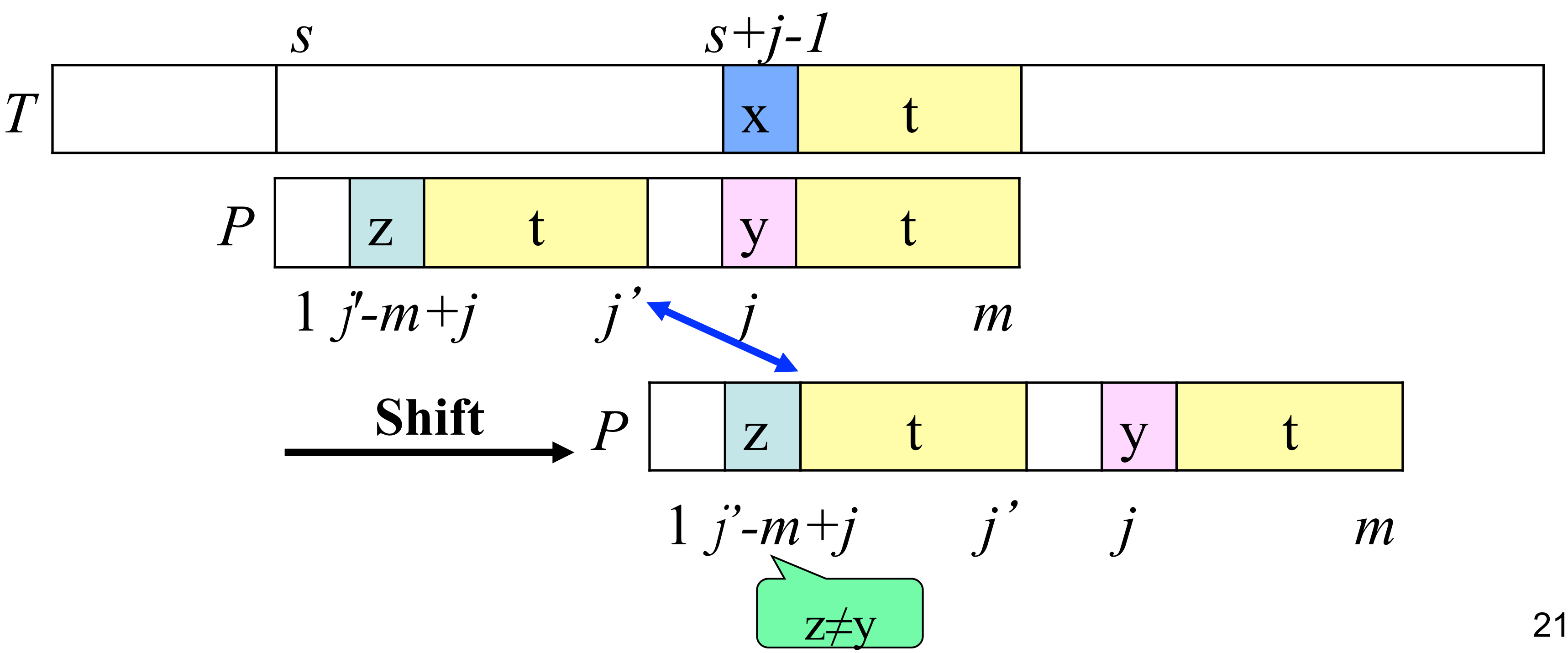
Good Suffix Rule 1

- If a mismatch occurs in T_{s+j-1}, we match T_{s+j-1} with $P_{j'-m+j}$, where j' ($m-j+1 \leqq j' < m$) is the **largest position** such that

  (1) $\boldsymbol{P_{j+1,m}}$ **is a suffix of** $\boldsymbol{P_{1,j'}}$

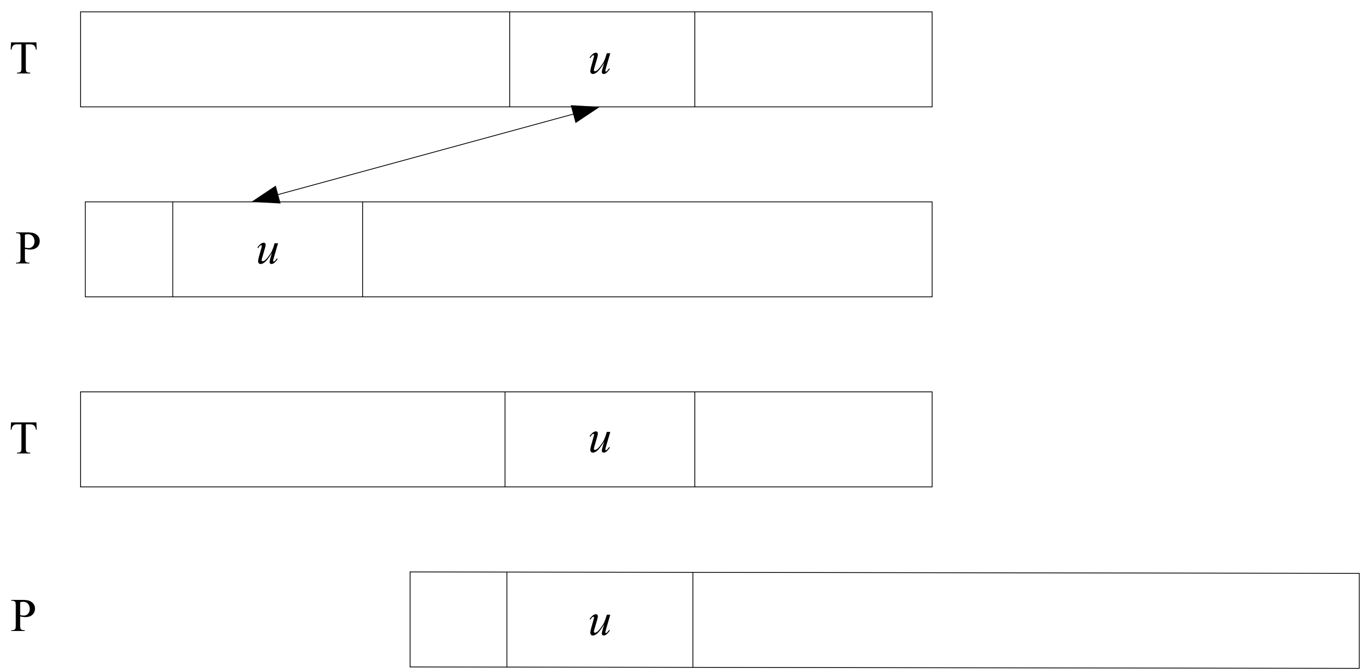
  (2) $\boldsymbol{P_{j'-(m-j)} \neq P_{j}}$.

- We can move the window at least ($m-j'$) position(s).

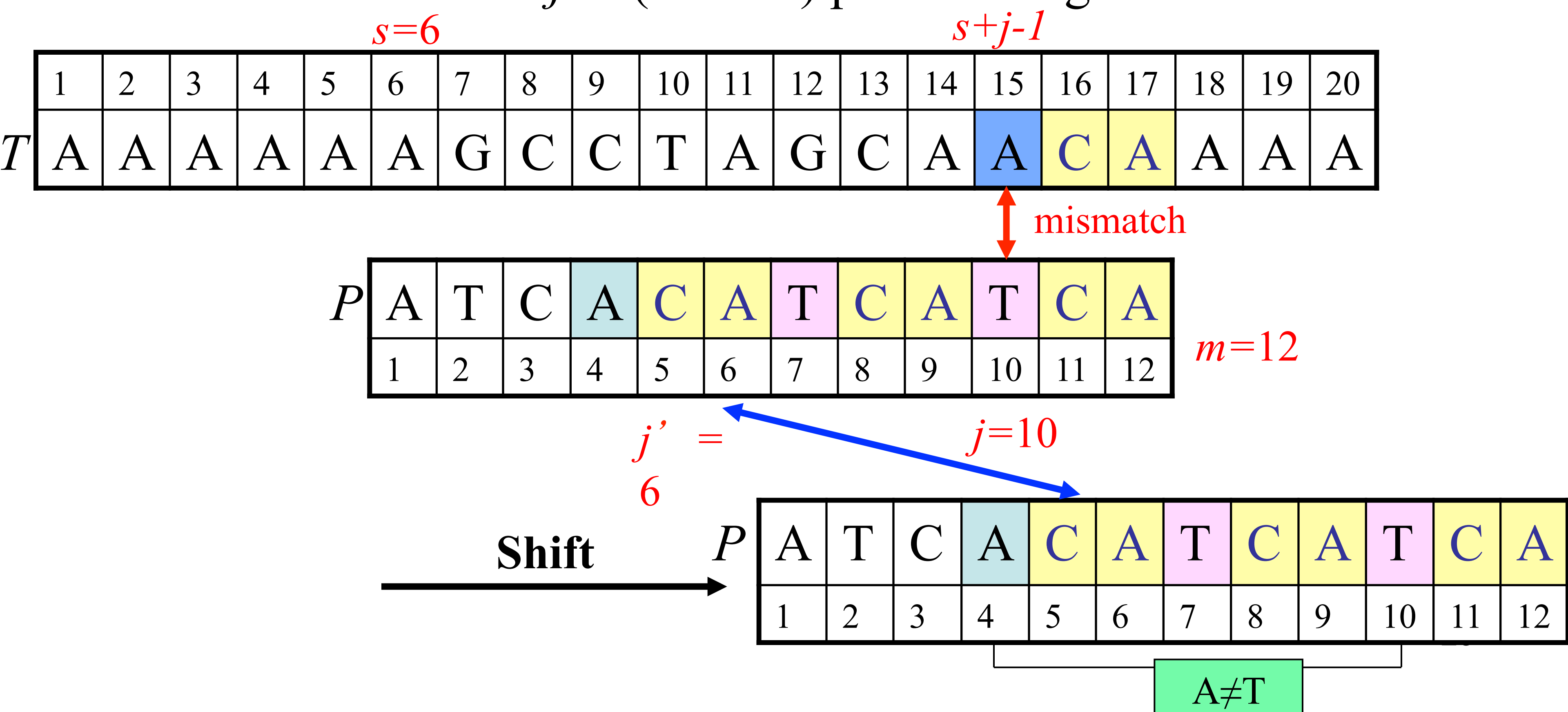


Rule: The Substring Matching Rule

- For any substring *u* in T, find a nearest *u* in P which is to the left of it. If such a *u* in P exists, move P; otherwise, we may define a new partial window.



- Ex: Suppose that P_1 is aligned to T_6 now. We compare pair-wise between P and T from right to left. Since $T_{16,17}$ = "CA" = $P_{11,12}$ and T_{15} = "A" $\neq P_{10}$ = "T". We find the substring "CA" in the left of P_{10} in P such that "CA" is the suffix of $P_{1,6}$ and the left character to this substring "CA" in P is not equal to P_{10} = "T". Therefore, we can move the window at least m-j' (12-6=6) positions right.

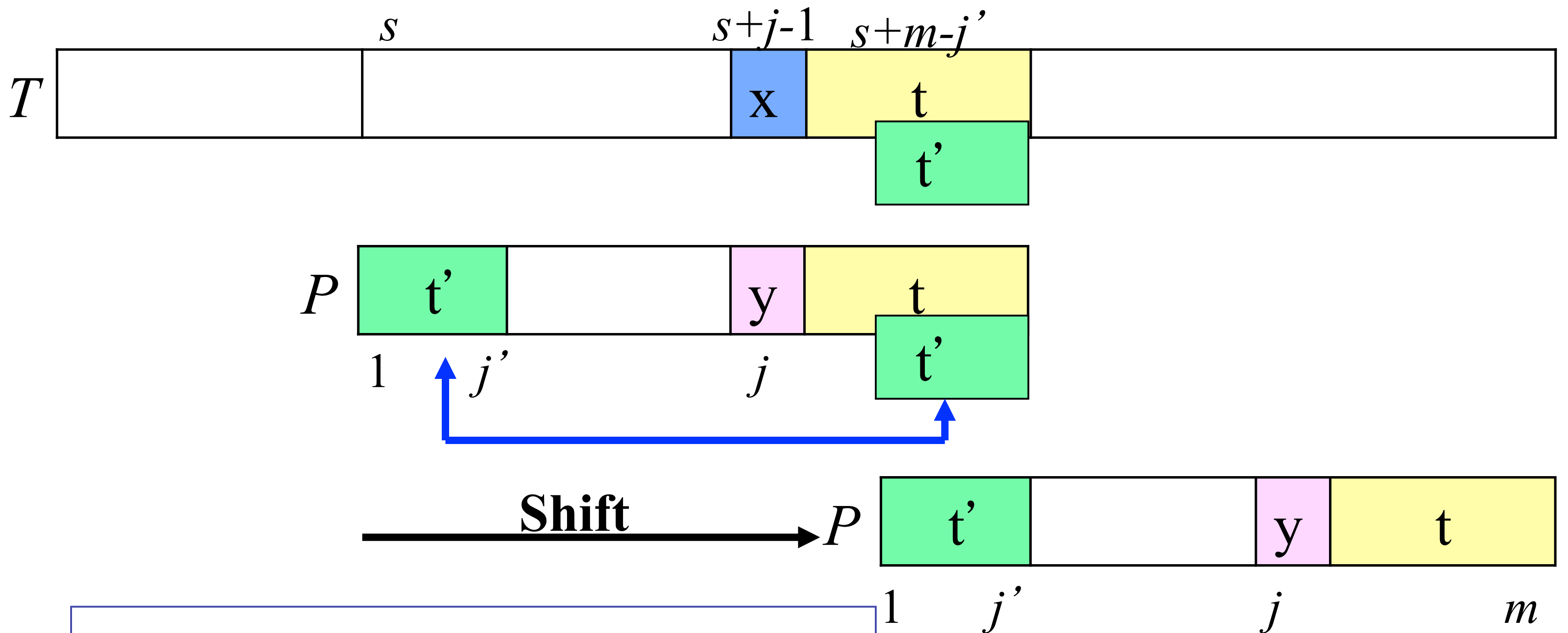


Good Suffix Rule 2

Good Suffix Rule 2 is used only when Good Suffix Rule 1 can not be used. That is, t does not appear in P(1, j). Thus, t is **unique** in P.

- If a mismatch occurs in T_{s+j-1}, we match $T_{s+m-j'}$ with P_1, where j' $(1 \leqq j' \leqq m-j)$ is **the largest position** such that

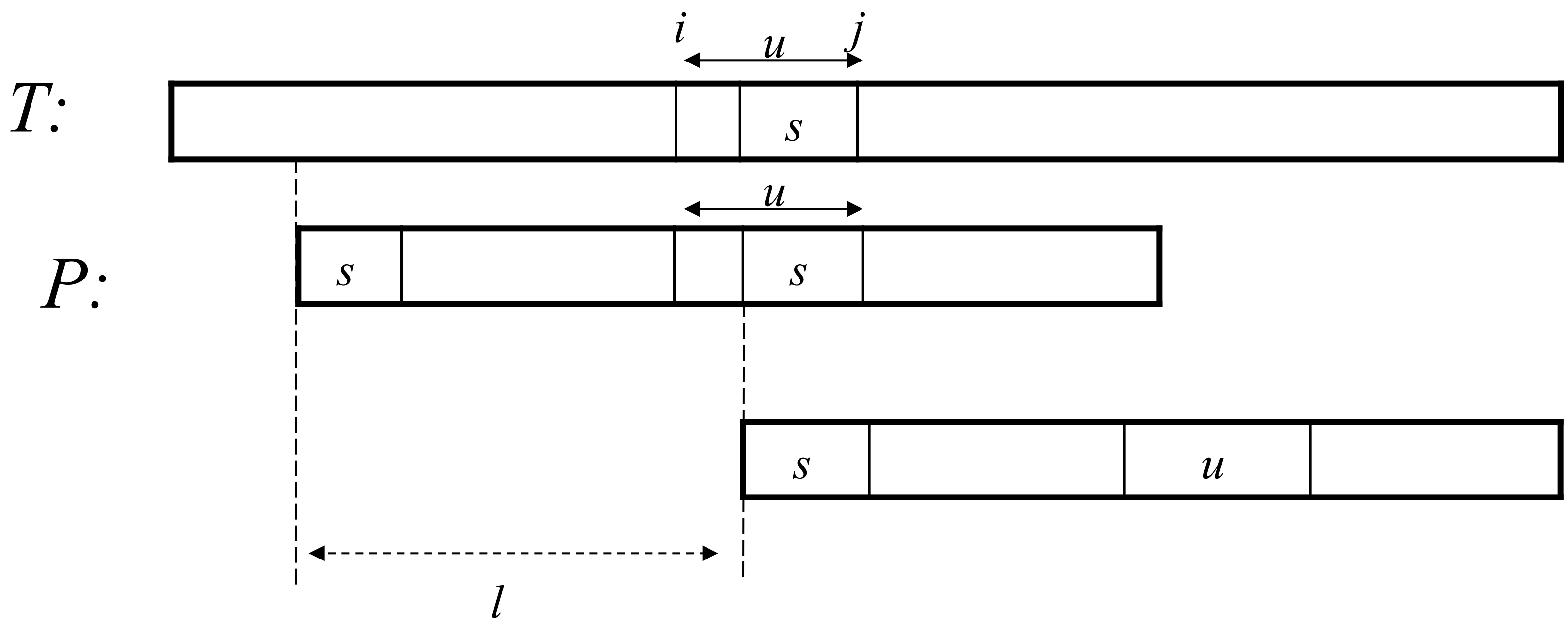
$P_{1,j'}$ is a suffix of $P_{j+1,m}$.



P.S. : t' is suffix of substring t.

Rule: Unique Substring Rule

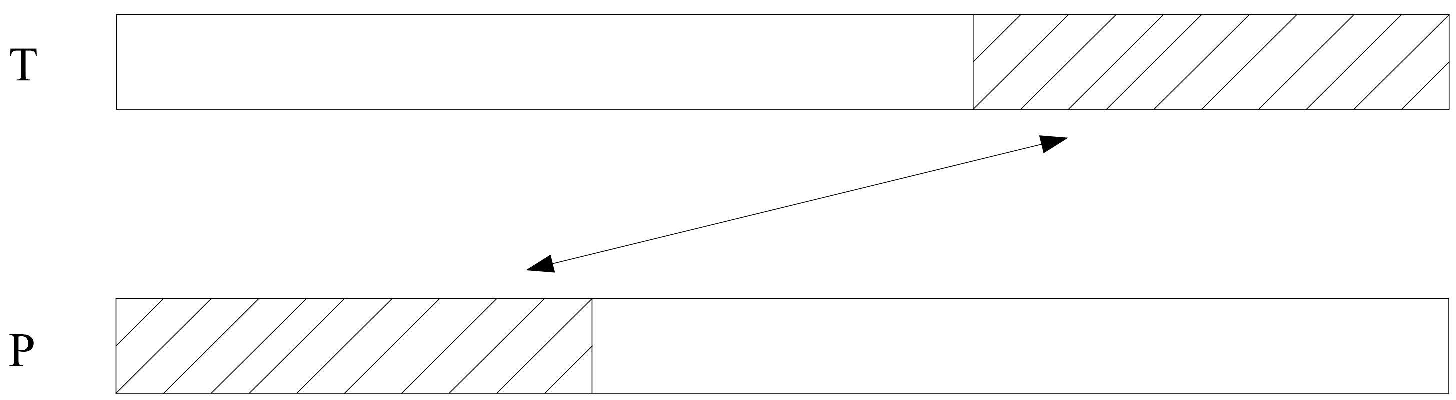
- The substring u appears in P exactly once.
- If the substring u matches with $T_{i,j}$, no matter whether a mismatch occurs in some position of P or not, we can slide the window by l.



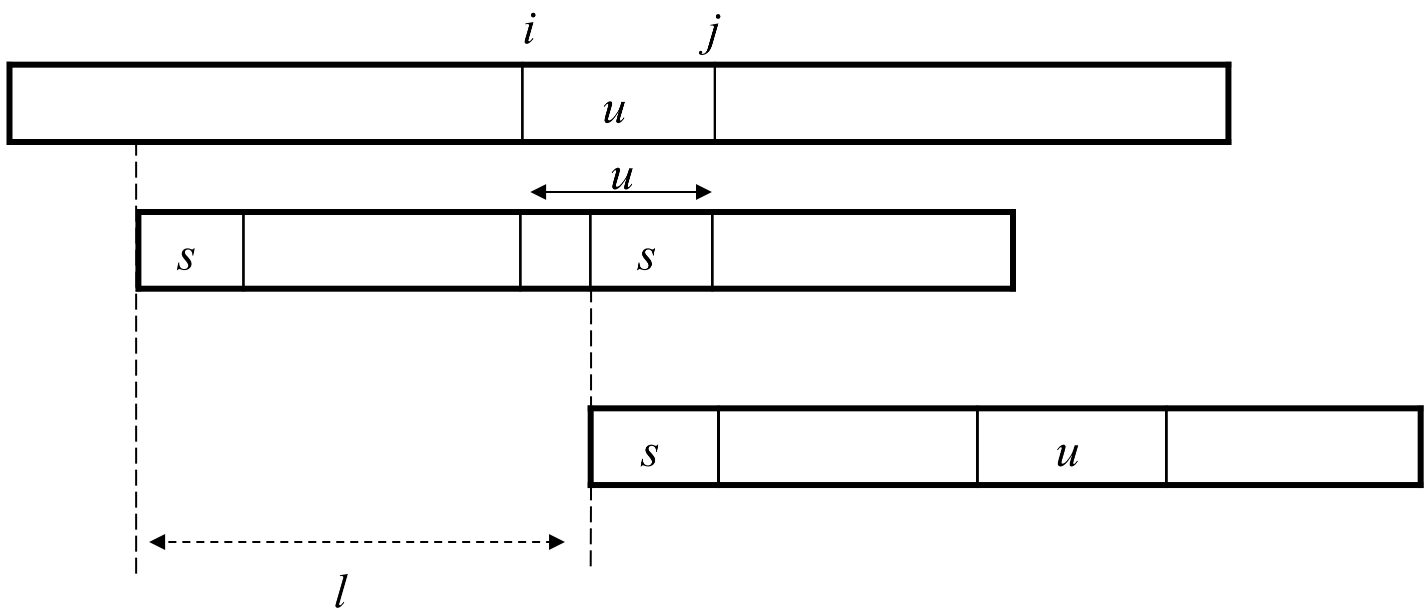
The string s is the longest prefix of P which equals to a suffix of u.

The Suffix to Prefix Rule

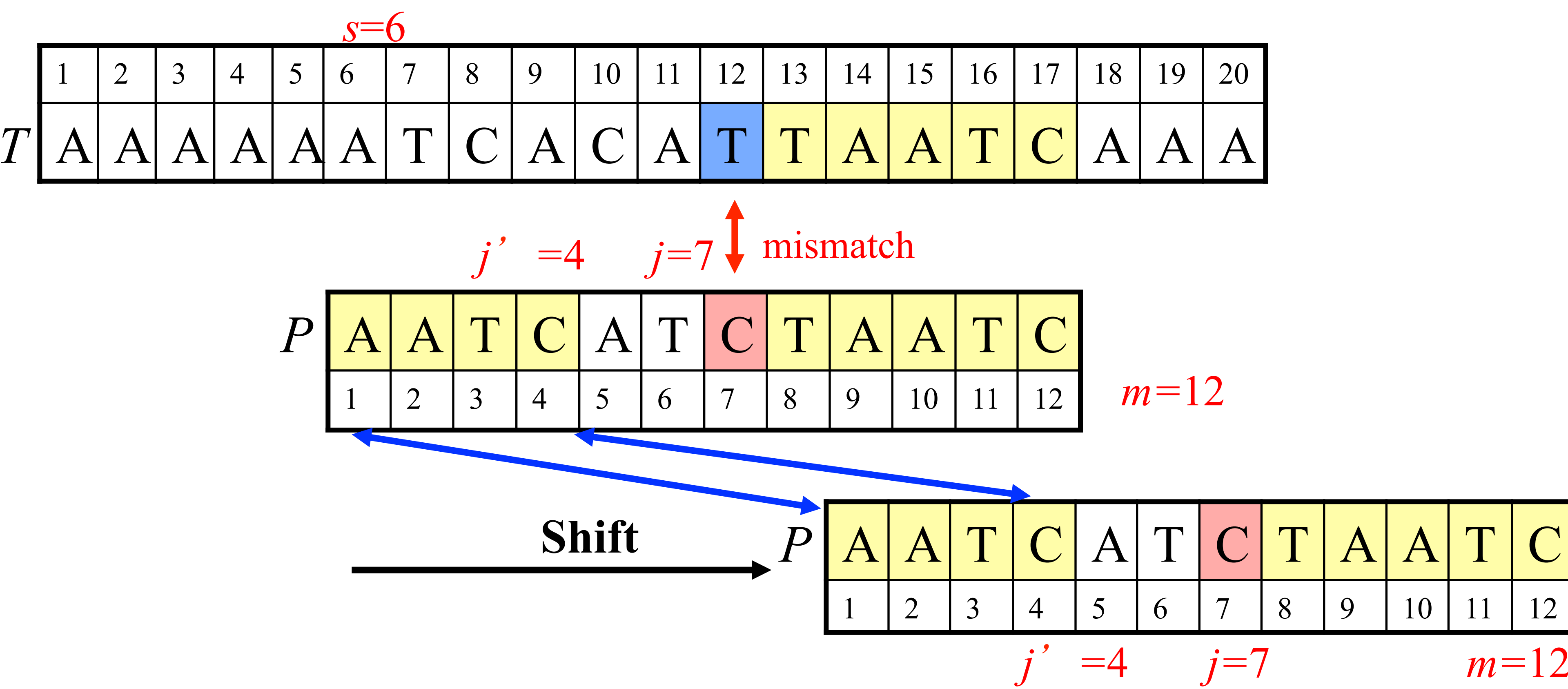
- For a window to have any chance to match a pattern, in some way, there must be a suffix of the window which is equal to a prefix of the pattern.



- Note that the above rule also uses Rule 1.
- It should also be noted that the unique substring is the shorter and the more right-sided the better.
- A short *u* guarantees a short (or even empty) *s* which is desirable.



- Ex: Suppose that P_1 is aligned to T_6 now. We compare pair-wise between P and T from right to left. Since $T_{12} \neq P_7$ and there is no substring $P_{8,12}$ in left of P_8 to exactly match $T_{13,17}$. We find a longest suffix "AATC" of substring $T_{13,17}$, the longest suffix is also prefix of P. We shift the window such that the last character of prefix substring to match the last character of the suffix substring. Therefore, we can shift at least 12-4=8 positions.



- Let ***Bc*(*a*)** be the rightmost position of *a* in *P*. The function will be used for applying ***bad character rule***.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A

Σ	A	C	G	T
B	12	11	0	10

- We can move our pattern right $\boldsymbol{j\text{-}B(T_{s+j-1})}$ position by above ***Bc*** function.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
T	A	G	C	T	A	G	C	C	T	G	C	A	C	G	T	A	C	A

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A

Move
10-$B(G)$ = 10 positions

Let ***Gs*(*j*)** be **the largest number of shifts** by ***good suffix rule*** when a mismatch occurs for comparing P_j with some character in *T*.

- $\boldsymbol{gs_1(j)}$ be the largest $\boldsymbol{k}$ such that $\boldsymbol{P_{j+1,m}}$ **is a suffix of** $\boldsymbol{P_{1,k}}$ **and** $\boldsymbol{P_{k-m+j} \neq P_j}$, where $m\text{-}j+1 \leq k<m$; 0 if there is no such k.

  (gs_1 is for Good Suffix Rule 1)

- $\boldsymbol{gs_2(j)}$ be the largest $\boldsymbol{k}$ such that $\boldsymbol{P_{1,k}}$ **is a suffix of** $\boldsymbol{P_{j+1,m}}$, where $1 \leq k \leq m\text{-}j$; 0 if there is no such k.

  (gs_2 is for Good Suffix Rule 2.)

- $\boldsymbol{Gs(j) = m - \max\{gs_1, gs_2\}}$, if $j = m$, $Gs(j)$=1.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	**A**	**T**	**C**	**A**	**C**	**A**	**T**	**C**	**A**	**T**	**C**	**A**
$\boldsymbol{gs_1}$	0	0	0	0	0	0	9	0	0	6	1	0
$\boldsymbol{gs_2}$	4	4	4	4	4	4	4	4	1	1	1	0
Gs	8	8	8	8	8	8	3	8	11	6	11	1

$gs_1(7)$=9

$\because$ $P_{8,12}$ is a suffix of $P_{1,9}$

and $P_4 \neq P_7$

$gs_2(7)$=4

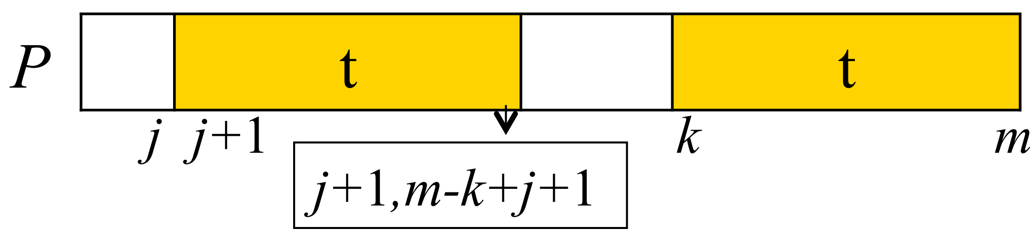
$\because P_{1,4}$ is a suffix of $P_{8,12}$

How do we obtain gs_1 and gs_2?

In the following, we shall show that by constructing the **Suffix Function**, we can kill two birds with one arrow.

Suffix function f'

- For $1 \leqq j \leqq m$-1, let the suffix function $f'(j)$ for $\boldsymbol{P_j}$ be the **smallest** $\boldsymbol{k}$ such that $\boldsymbol{P_{k,m} = P_{j+1,m-k+j+1}}$; ($\boldsymbol{j+2} \leqq k \leqq m$)
  - If there is no such k, we set $f' = m+1$.
  - If $j=m$, we set $f'(m)=m+2$.



- Ex:

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14

- $f'(4)=8$, it means that $P_{f'(4),m} = P_{8,12} = P_{5,9} = P_{4+1,4+1+m-f'(4)}$
- Since there is no k for $13=j+2 \leqq k \leqq 12$, we set $f'(11)=13$.

Suppose that the Suffix is obtained. How can we use it to obtain gs_1 and gs_2?

gs_1 can be obtained by scanning the Suffix function from right to left.

Example

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
T	G	A	T	C	G	A	T	C	A	A	T	C	A	T	C	A	C	A	T	G	A	T	C	A

P	A	T	C	A	C	A	T	C	A	T	C	A
	1	2	3	4	5	6	7	8	9	10	11	12

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14

Example

As for Good Suffix Rule 2, it is relatively easier.

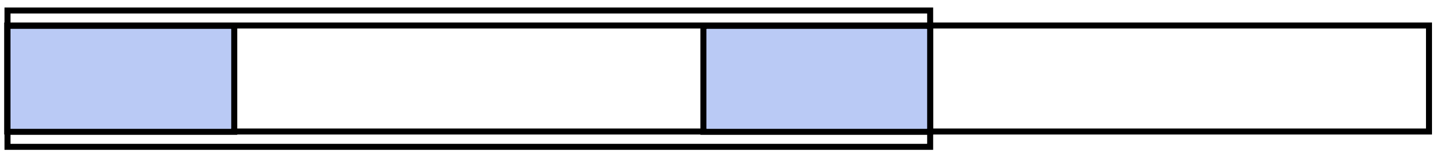
j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14

Question: How can we construct the Suffix function?

To explain this, let us go back to the prefix function used in the KMP Algorithm.

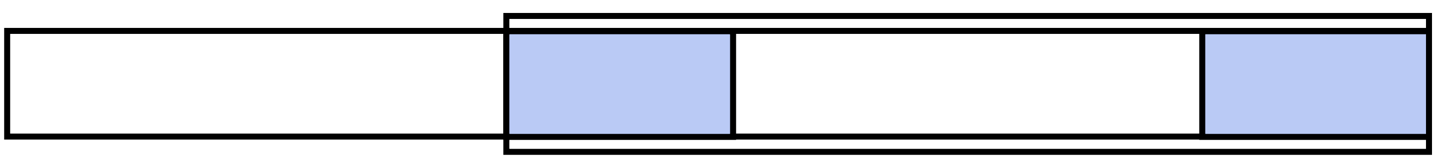
The following figure illustrates the prefix function in the KMP Algorithm.

P



The following figure illustrates the suffix function of the BM Algorithm.

P



We now can see that actually the suffix function is the same as the prefix. The only difference is now we consider a suffix. Thus, the recursive formula for the prefix function in KMP Algorithm can be slightly modified for the suffix function in BM Algorithm.

- The formula of suffix function f' as follows :

Let $f'^{x}(y) = f'(f'^{x-1}(y))$ for $x > 1$ and $f'^{1}(y) = f'(y)$

$$f'(j) = \begin{cases} m+2, & \text{if } j = m \\ f'^{k}(j+1)-1, & \text{if } 1 \le j \le m\text{-}1 \text{ and there exists the smallest} \\ & k \ge 1 \text{ such that } P_{j+1} = P_{f'^{k}(j+1)-1}; \\ m+1, & \text{otherwise} \end{cases}$$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'												14

j=m=12, f' =m+2=14

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'											13	14

No *k* satisfies

$P_{j+1}=P_{f'^{k}(j+1)-1}$,

f' $=m+1=12+1=13$

$k=1 \rightarrow$

$P_{12} \neq P_{13}$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'										13	13	14

No k satisfies
$P_{j+1}=P_{f'^{k}(j+1)-1}$,
$f' = m+1=12+1=13$

$k=1 \rightarrow$

$P_{11} \neq P_{12}$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'									13	13	13	14

No k satisfies
$P_{j+1}=P_{f'^{k}(j+1)-1}$,
$f' = m+1=12+1=13$

$k=1 \rightarrow$

$P_{10} \neq P_{12}$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'								12	13	13	13	14

$\because P_{j+1} = P_{f'(j+1)-1} \Rightarrow P_9 = P_{12}$,

$f'(j) = f'(j+1) - 1 = 13 - 1 = 12$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'							11	12	13	13	13	14

$\because P_{j+1} = P_{f'(j+1)-1} \Rightarrow P_8 = P_{11}$,

$f'(j) = f'(j+1) - 1 = 12 - 1 = 11$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'				8	9	10	11	12	13	13	13	14

$\because P_{j+1} = P_{f'^{1}(j+1)-1} \Rightarrow P_5 = P_8$,

$f' = f'(j+1) - 1 = 9 - 1 = 8$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'			12	8	9	10	11	12	13	13	13	14

$\because P_{j+1} = P_{f'^{3}(j+1)-1} \Rightarrow P_4 = P_{f'^{3}(4)-1} = P_{12}$,

$f' = f'^{3}(j+1) - 1 = 13 - 1 = 12$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'		11	12	8	9	10	11	12	13	13	13	14

$\because P_{j+1} = P_{f'(j+1)-1} \Rightarrow P_3 = P_{f'(3)-1} = P_{11}$,
$f' = f'(j+1) - 1 = 12 - 1 = 11$

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14

$\because P_{j+1} = P_{f'(j+1)-1} \Rightarrow P_2 = P_{f'(2)-1} = P_{10}$,
$f' = f'(j+1) - 1 = 11 - 1 = 10$

- Let $G'(j)$, $1 \leqq j \leqq m$,to be the largest number of shifts by good suffix rules.

- First, we set $G'(j)$ to zeros as their initializations.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	0	0	0	0	0	0	0	0	0	0	0	0

- **Step1:** We scan from right to left and $gs_1(j)$ is determined during the scanning, then $gs_1(j) >= gs_2(j)$

> **Observe:**
> If $P_j = P_4 \neq P_7 = P_{f'(j)-1}$, we know $\mathbf{gs_1(f'(j)-1)=m+j-f'(j)+1}=9$.
> **If $t = f'(j)-1 \leqq m$ and $P_j \neq P_t$, $G'(t) = m - gs_1(f'(j)-1) = f'(j) - 1 - j$.**
> **$f'^{(k)}(x) = f'^{(k-1)}(f'(x) - 1)$, $k \geq 2$**

- When j=12, t=13. $t > m$.
- When j=11, t=12. Since P_{11}= 'C' $\neq$ 'A' $= P_{12}$,
  $G'(t) = m - \max\{gs_1(t), gs_2(t)\} = m - \underline{gs_1(t)}$
  $= f'(j) - 1 - j$
  $\Rightarrow G'(12)$=13-1-11= 1.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	0	0	0	0	0	0	0	0	0	0	0	1

If $t = f'(j)-1 \leqq m$ and $P_j \neq P_t$, $G'(t)=f'(j) - 1 - j$.
$f'^{(k)}(x)=f'^{(k-1)}(f'(x) - 1)$, $k \geq 2$

- When j=10, t=12. Since P_{10}= 'T' ≠ 'A' =P_{12}, $G'(12) \neq 0$.
- When j=9, t=12. P_9= 'A' =P_{12}.
- When j=8, t=11. P_8= 'C' =P_{11}.
- When j=7, t=10. P_7= 'T' =P_{10}
- When j=6, t=9. P_6 = 'A' =P_9
- When j=5, t=8. P_5 = 'C' =P_8
- When j=4, t=7. Since P_4= 'A' ≠ P_7 = 'T', $G'(7) = 8 - 1 - 4 = 3$

Besides, $t = f'^{(2)}(4) - 1 = f'(f'(4) - 1) - 1 = 10$. Since P_4 = 'A' ≠ P_{10} = 'T', $G'(10) = f'(7) - 1 - j = 11 - 1 - 4 = 6$.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	0	0	0	0	0	0	3	0	0	6	0	1

If $t = f'(j)-1 \leqq m$ and $P_j \neq P_t$, $G'(t)=f'(j) - 1 - j$.
$f'^{(k)}(x)=f'^{(k-1)}(f'(x) - 1)$, $k \geq 2$

- When j=3, t=11. P_3= 'C' =P_{11}.
- When j=2, t=10. P_2= 'T' =P_{10}
- When j=1, t=9. P_1= 'A' =P_9.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	0	0	0	0	0	0	3	0	6	0	0	1

- By the above discussion, we can obtain the values using the Good Suffix Rule 1 by scanning the pattern from right to left.

- **Step2:** Continuously, we will try to obtain the values using ***Good Suffix Rule 2*** and those values are still zeros now and scan from left to right.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	0	0	0	0	0	0	3	0	0	6	0	1

- Let k' be the **smallest** $\boldsymbol{k}$ in $\{1,\ldots,m\}$ such that $\boldsymbol{P_{f'^{(k)}(1)-1} = P_1}$ and $f'^{(k)}(1)-1 <= m$.

Observe:
$\because P_{1,4}=P_{9,12}$, $\therefore gs_2(j)=m-(f'(1)-1)+1=4$, where $1 \leqq j \leqq f'^{(k')}(1)-2$.

- If $G'(j)$ is not determined in the first scan and $1<=j<=f'^{(k')}(1)-2$, thus, in the second scan, we set $G'(j)=m - \max\{gs_1(j), gs_2(j)\}= m - gs_2(j)=f'^{(k')}(1) - 2$. If no such k exists, set each undetermined value of G to m in the second scan.

- ***k*=1=*k'***, since $P_{f'(1)-1}=P_9=$ "A" $=P_1$, we set $G'(j)=f'(1)-2$ for j=1,2,3,4,5,6,8.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	8	8	8	8	8	8	3	8	0	6	0	1

- Let z be $f'^{(k')}(1)$-2. Let k'' be the **largest value *k*** such that $f'^{(k)}(z)-1 <= m$.
- Then we set $G'(j) = m - gs_2(j) = m - (m - f'^{(i)}(z) - 1) = f'^{(i)}(z) - 1$, where $1<=i<=k''$ and $f'^{(i-1)}(z) < j <= f'^{(i)}(z)-1$ and $f'^{(0)}(z) = z$.

- For example, z=8 :
  - $k=1, f'^{(1)}(8)-1=11 \leqq m=12$
  - $k=2, f'^{(2)}(8)-1=12 \leqq m=12$ **=> *k''=2***
  - $i=1, f'^{(0)}(8)-1 = 7 < \boldsymbol{j} \leqq f'^{(1)}(8)-1=11.$
  - $i=2, f'^{(1)}(8)-1 = 11 < \boldsymbol{j} \leqq f'^{(2)}(8)-1=12.$
  - We set $G(9)$ and $G(11) = f'^{(1)}(8) - 1 = 12-1 = 11$.

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	8	8	8	8	8	8	3	8	11	6	11	1

We essentially have to decide the maximum number of steps. We can move the window right when a mismatch occurs. This is decided by the following function:

$$\max\{G'(j), j\text{-}B(T_{s+j-1})\}$$

Example

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
T	G	A	T	C	G	A	T	C	A	C	A	T	A	T	C	A	C	A	T	C	A	T	C	A

mismatch

P	A	T	C	A	C	A	T	C	A	T	C	A
	1	2	3	4	5	6	7	8	9	10	11	12

Shift

P	A	T	C	A	C	A	T	C	A	T	C	A
	1	2	3	4	5	6	7	8	9	10	11	12

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	8	8	8	8	8	8	3	8	11	6	11	1

Σ	A	C	G	T
B	12	11	0	10

We compare *T* and *P* from right to left. Since T_{12}= "T" $\neq P_{12}$= "A", the largest movement = max{$G'(j), j-B(T_{s+j-1})$} = max{$G'(12), 12-B(T_{12})$}= max{1,12-10} = 2.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
T	G	A	T	C	G	A	T	C	A	C	A	T	A	T	C	A	C	A	T	C	A	T	C	A

mismatch

P	A	T	C	A	C	A	T	C	A	T	C	A
	1	2	3	4	5	6	7	8	9	10	11	12

Shift →

P	A	T	C	A	C	A	T	C	A	T	C	A
	1	2	3	4	5	6	7	8	9	10	11	12

j	1	2	3	4	5	6	7	8	9	10	11	12
P	A	T	C	A	C	A	T	C	A	T	C	A
f'	10	11	12	8	9	10	11	12	13	13	13	14
G'	8	8	8	8	8	8	3	8	11	6	11	1

Σ	A	C	G	T
B	12	11	0	10

After moving, we compare *T* and *P* from right to left. Since T_{14}= "T" $\neq P_{12}$= "A", the largest movement = max{$G'(j)$, j-$B(Ts+j-1)$} = max{$G'(12)$, 12-$B(T_{14})$}
= max{1,12-10} = 2.

Time Complexity

- The preprocessing phase in $O(m+\Sigma)$ time and space complexity and searching phase in $\boldsymbol{O}(mn)$ time complexity.

- The worst case time complexity for the ***Boyer-Moore*** method would be $O((n-m+1)m)$.

- It was proved that this algorithm has $O(m)$ comparisons when P is not in T. However, this algorithm has $O(mn)$ comparisons when P is in T.

Reference

- **Algorithms for finding patterns in strings** , AHO, A.V., **Handbook of Theoretical Computer Science** ,Volume A ,Chapter 5 Elsevier , Amsterdam , 1990, pp. 255-300.
- **Computer algorithms: string pattern matching strategies** , Jun-ichi, A. , IEEE Computer Society Press , 1994.
- **Computer Algorithms: Introduction to Design and Analysis** , BAASE, S. and VAN GELDER, A. , **Addison-Wesley Publishing Company** , Chapter 11 , 1999.
- **Indexing and Searching** , BAEZA-YATES, R. , NAVARRO, G. and RIBEIRO-NETO, B. , **Modern Information Retrieval** , Chapter 8 , 1999 , pp. 191-228.
- **Éléments d'algorithmique** , BEAUQUIER, D., BERSTEL, J. and CHRÉTIENNE, P., Masson Paris , Chapter 10 , 1992 , pp. 337-377.
- **A fast string searching algorithm ,** BOYER R.S. and MOORE J.S. , **Communications of the ACM** , Vol 20 , 1977 , pp. 762-772 .
- **Tight bounds on the complexity of the Boyer-Moore pattern matching algorithm** , COLE, R., **SIAM Journal on Computing** , Vol 23 , 1994 , pp. 1075-1091.
- **Introduction to Algorithms** , CORMEN, T.H. , LEISERSON, C.E. and RIVEST, R.L. , **The MIT Press** , Chapter 34 , 1990 , pp. 853-885.
- **Off-line serial exact string searching** , CROCHEMORE, M. , **Pattern Matching Algorithms** , Chapter 1 , 1997 , pp 1-53
- **Pattern Matching in Strings** , CROCHEMORE, M. and HANCART, C. , **Algorithms and Theory of Computation Handbook** , Chapter 11, 1999 , pp. 11-1-11-28.

- **Pattern matching and text compression algorithms** , CROCHEMORE, M. and LECROQ, T. , **CRC Computer Science and Engineering Handbook** , Chapter 8 , 1996 , pp. 162-202.
- **Text Algorithms**, CROCHEMORE, M. and RYTTER, W., **Oxford University Press** , 1994.
- **Handbook of Algorithms and Data Structures in Pascal and C**, GONNET, G.H. and BAEZA-YATES, R.A. , **Addison-Wesley Publishing Company** , Chapter 7 , 1991 , pp. 251-288,.
- **Data Structures and Algorithms in JAVA** , GOODRICH, M.T. and TAMASSIA, R. , **John Wiley & Sons** , Chapter 11 , 1998 , pp. 441-467.
- **Algorithms on strings, trees** , GUSFIELD, D. , **Cambridge University Press** , 1997.
- **Analyse exacte et en moyenne d'algorithmes de recherche d'un motif dans un texte** , HANCART, C., University Paris 7, France , 1993.
- **Fast pattern matching in strings** , KNUTH, D.E. , MORRIS, J.H. and PRATT, V.R. , **SIAM Journal on Computing** , 1977 , pp.323-350.
- LECROQ, T., 1992, *Recherches de mot*,. Thesis, University of Orléans, France.
- **Experimental results on string matching algorithms** , LECROQ, T. , **Software - Practice & Experience** , Vol 25 , 1995 , pp. 727-765.
- **Algorithms** , SEDGEWICK, R. , **Addison-Wesley Publishing Company** , Chapter 19 , 1988 , pp. 277-292.
- **Algorithms in C** , SEDGEWICK, R. , **Addison-Wesley Publishing Company** , Chapter 19 , 1988.
- **String Searching Algorithms** , STEPHEN, G.A. , World Scientific , 1994.
- **Taxonomies and Toolkits of Regular Language Algorithms ,** WATSON, B.W. , **Eindhoven University of Technology** , 1995.
- **Algorithms & Data Structures** , WIRTH, N. , Prentice-Hall , Chapter 1 , 1986 , pp. 17-72.

Suffix trees and suffix arrays

String/Pattern Matching

- You are given a source string S.
- Answer queries of the form: is the string p_i a substring of S?
- Knuth-Morris-Pratt (KMP) string matching.
  - $O(|S| + | p_i |)$ time per query.
  - $O(n|S| + S_i | p_i |)$ time for n queries.
- Suffix tree solution.
  - $O(|S| + S_i | p_i |)$ time for n queries.

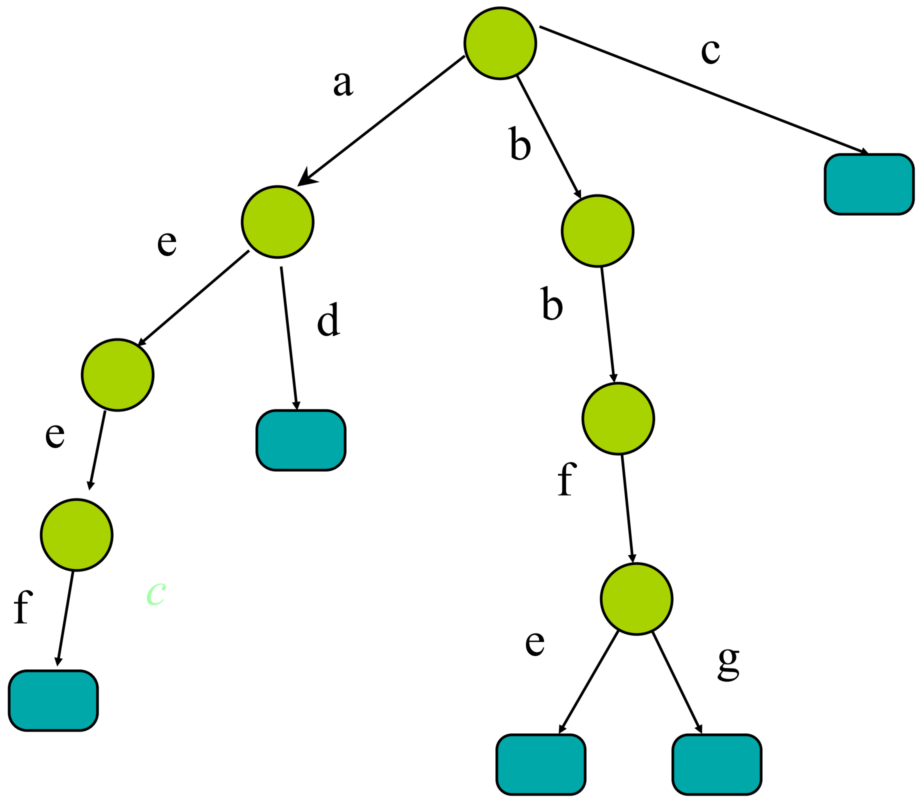
String/Pattern Matching

- KMP/BM preprocesses the query string p_i, whereas the suffix tree method preprocesses the source string S.

Trie

- A tree representing a set of strings.

{
aeef
ad
bbfe
bbfg
c }

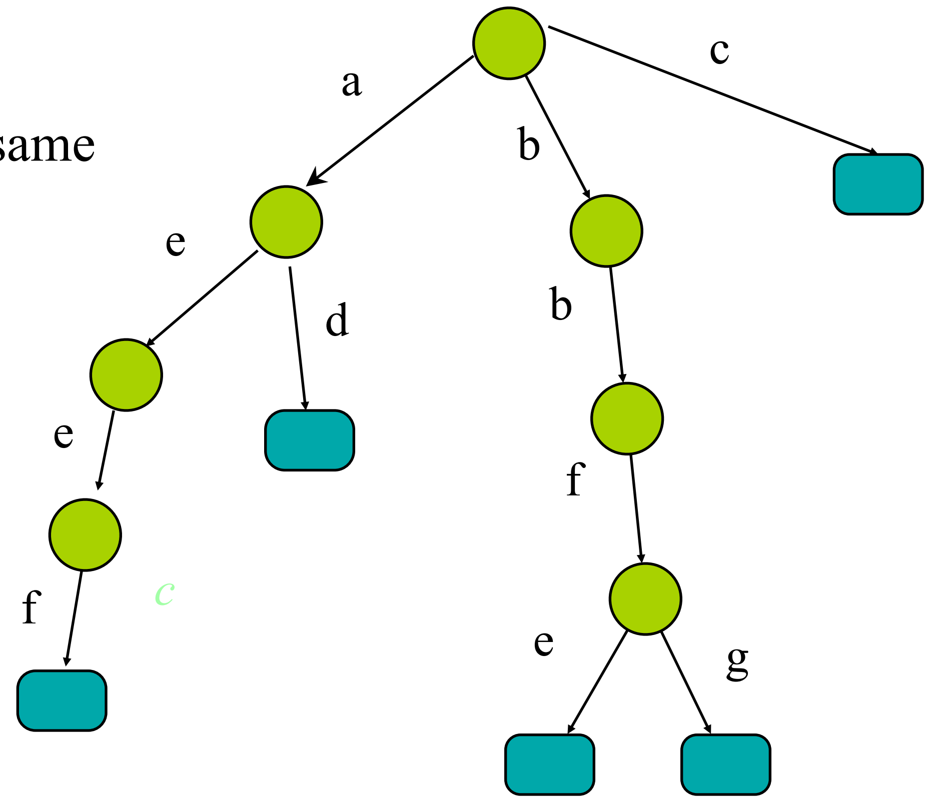


Trie (Cont)

- Assume no string is a prefix of another

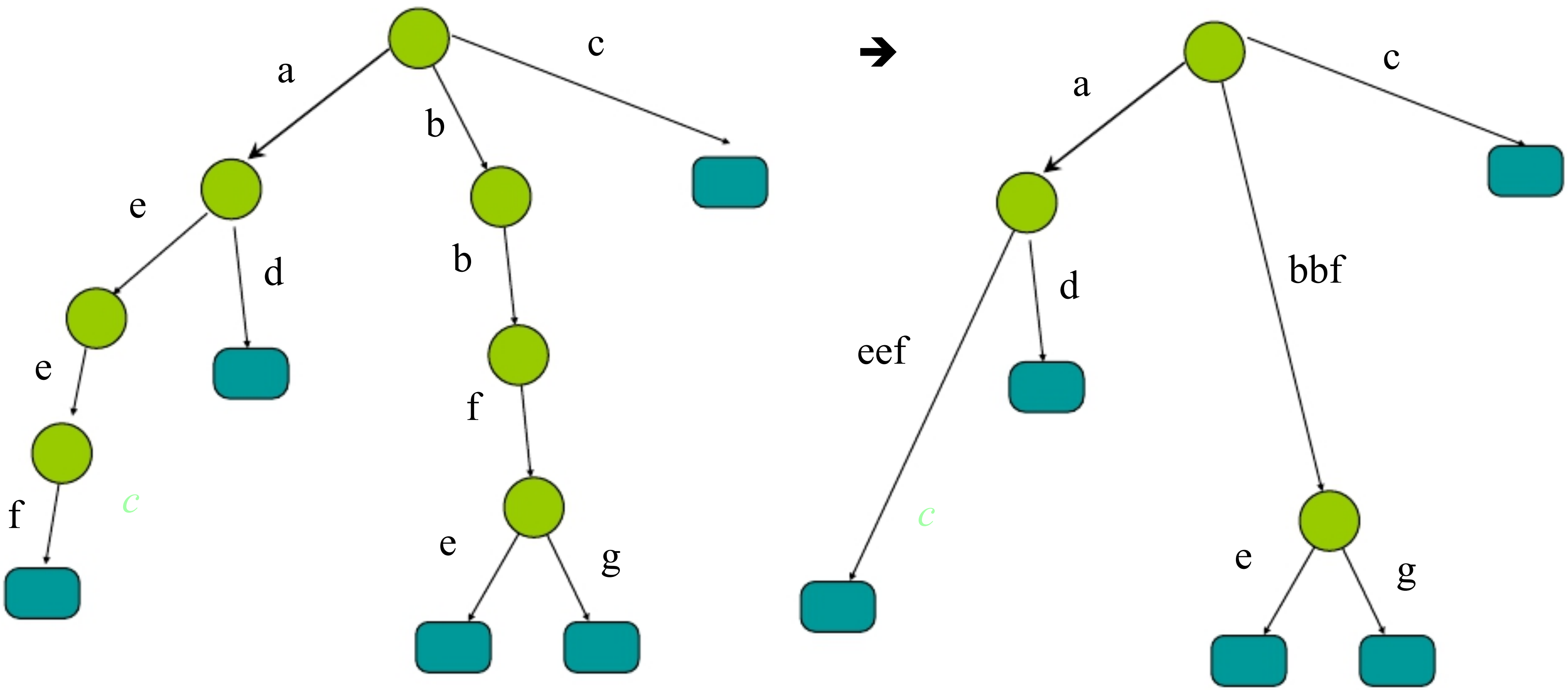
Each edge is labeled by a letter,
no two edges outgoing from the same
node are labeled the same.

Each string corresponds to a leaf.



Compressed Trie

- Compress unary nodes, label edges by strings



Suffix tree

Given a string s a suffix tree of s is a compressed trie of all suffixes of s

To make these suffixes prefix-free we add a special character, say $, at the end of s

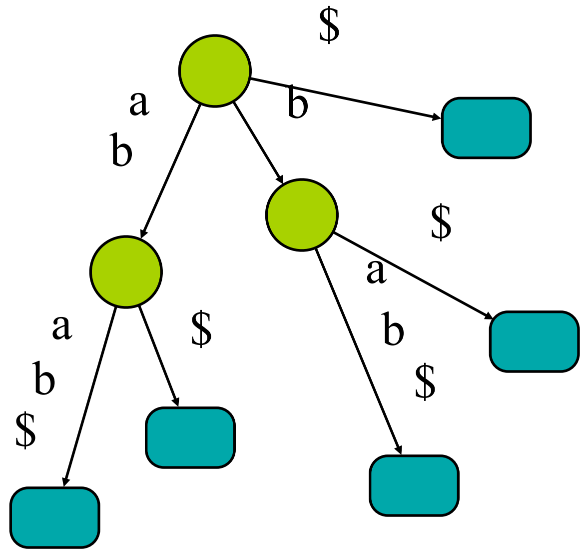
The suffix tree Tree(T) of T

- data structure suffix tree, Tree(T), is compacted trie that represents all the suffixes of string T
- linear size: |Tree(T)| = O(|T|)
- can be constructed in linear time O(|T|)
- has *myriad virtues* (A. Apostolico)
- is well-known: Google hits

Suffix tree (Example)

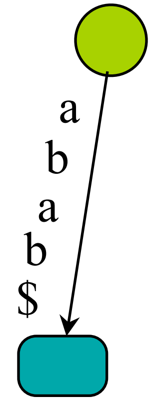
Let s=abab, a suffix tree of s is a compressed trie of all suffixes of s=abab$

{
$
b$
ab$
bab$
abab$ }

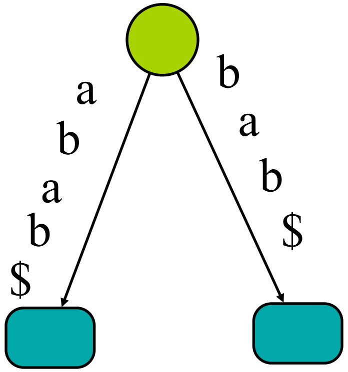


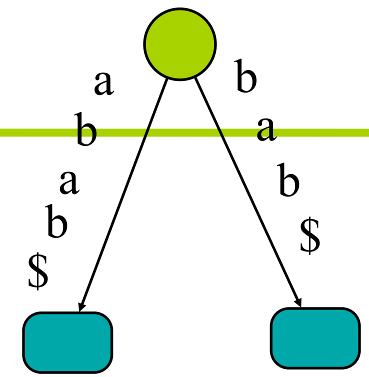
Trivial algorithm to build a Suffix tree

Put the largest suffix in

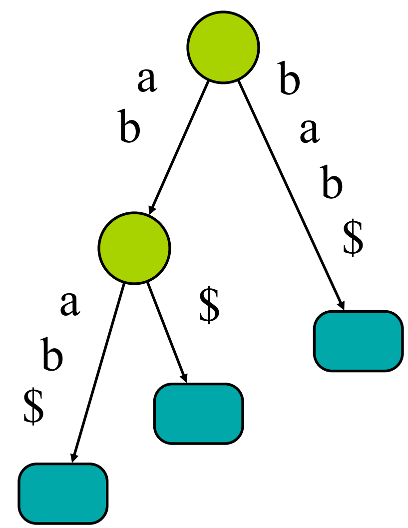


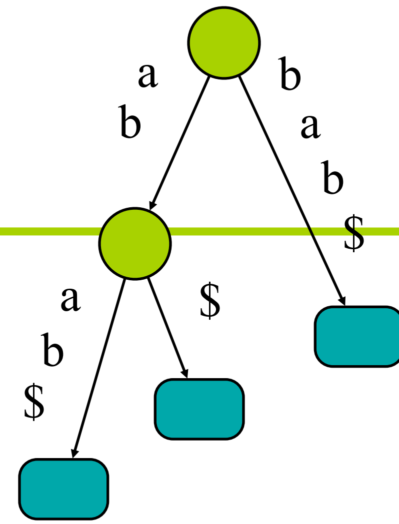
Put the suffix bab$ in



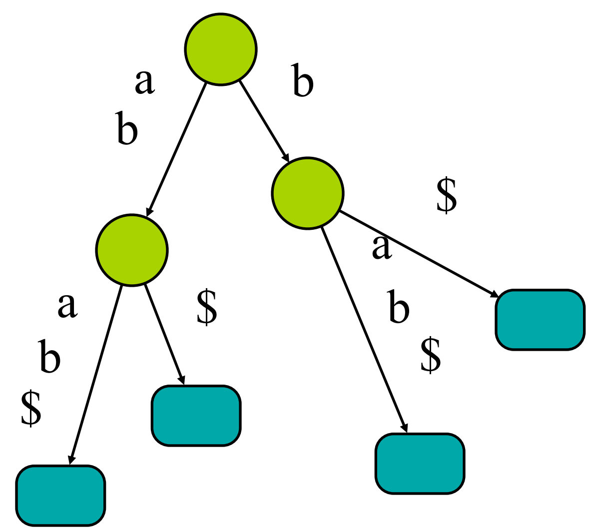


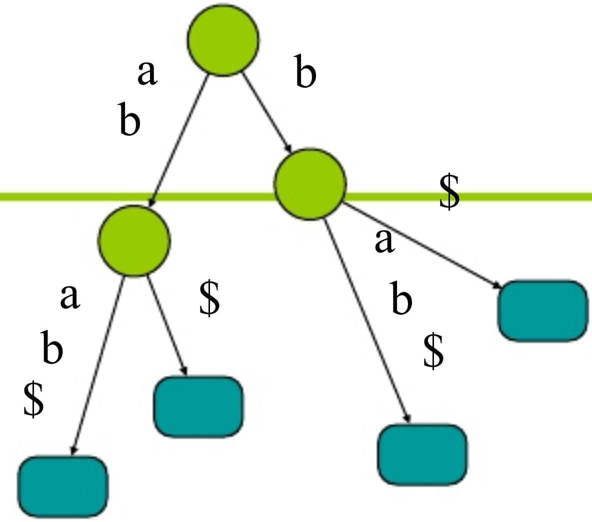
Put the suffix ab$ in



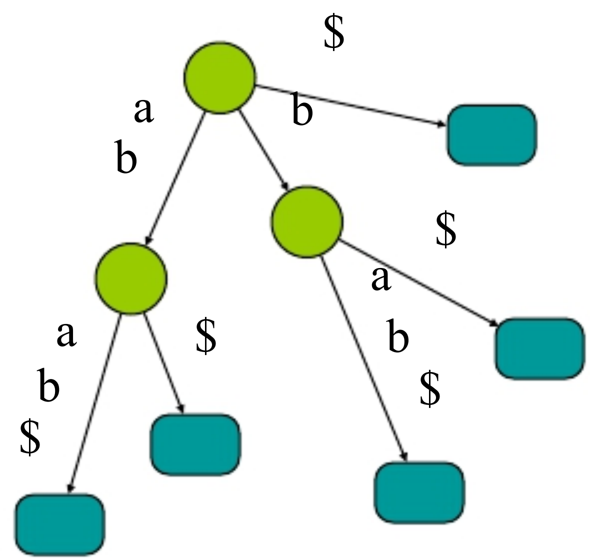


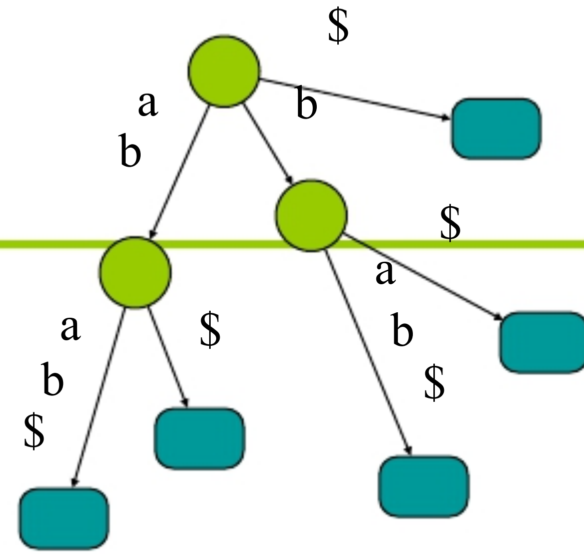
Put the suffix b$ in



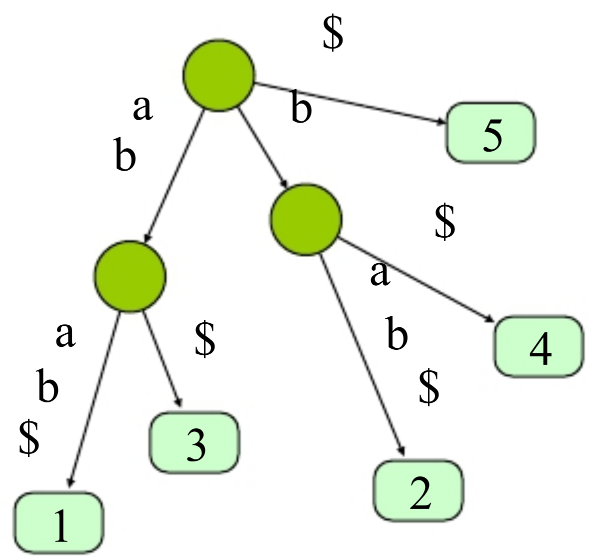


Put the suffix $ in





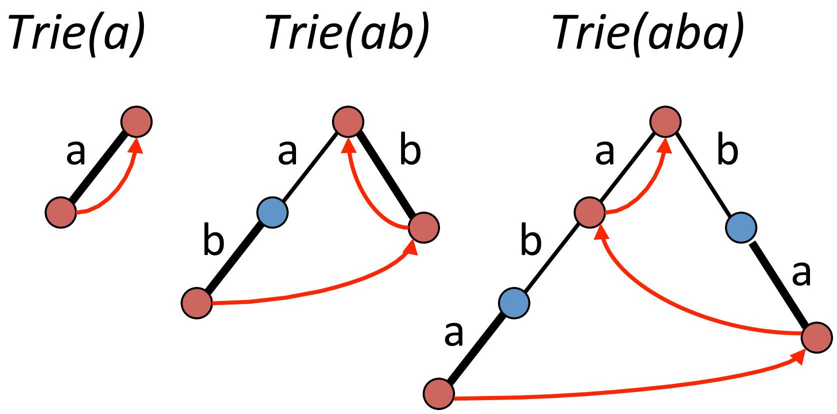
We will also label each leaf with the starting point of the corres. suffix.



On-line construction of Trie(T)

- $T = t_1 t_2 \ldots t_n\$$
- $P_i = t_1 t_2 \ldots t_i$ i:th prefix of *T*
- on-line idea: update *Trie(*P_i*)* to *Trie(*P_{i+1}*)*
- => very simple construction

Trie(abaab)

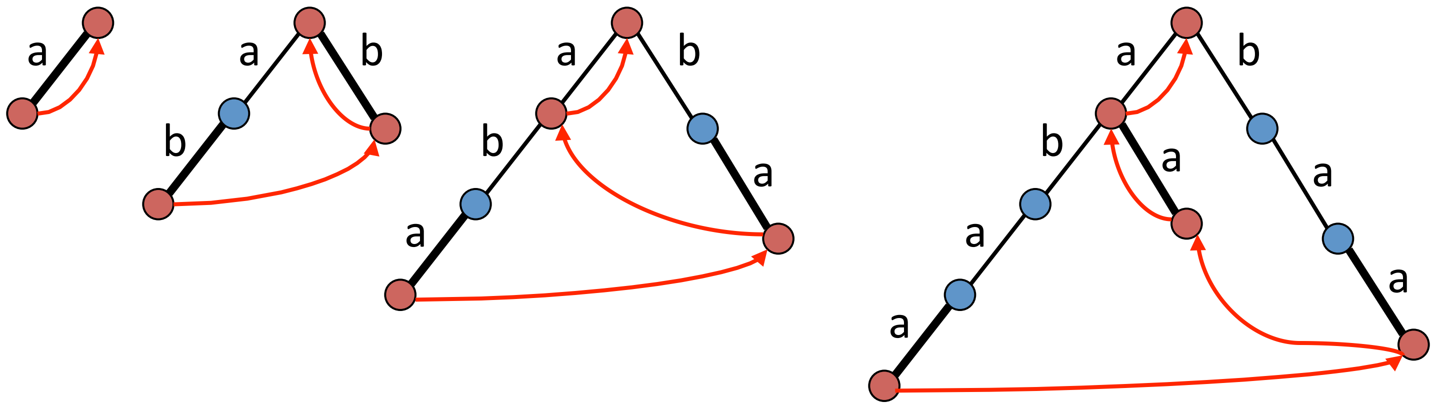


aba**a**
ba**a**
a**a**
ε**a**
ε

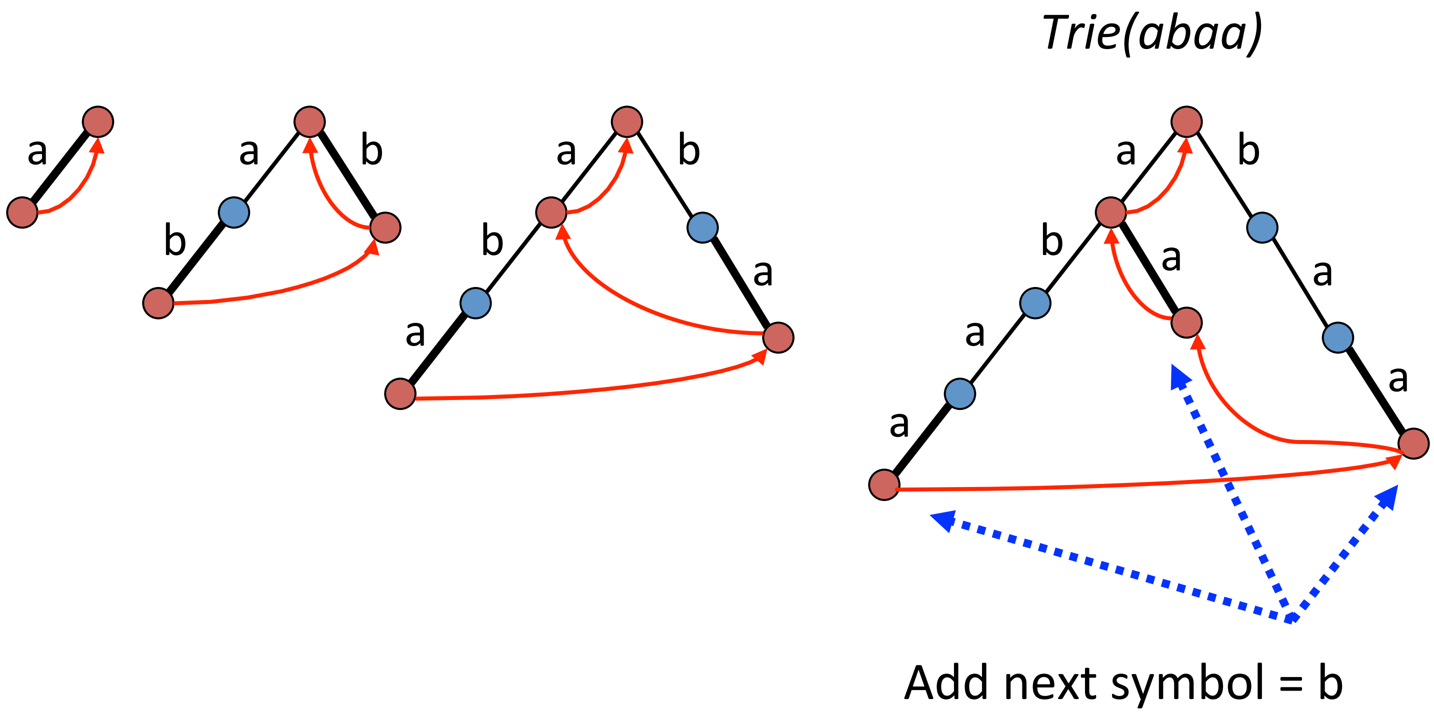
chain of links connects the end points of current suffixes

Trie(abaab)

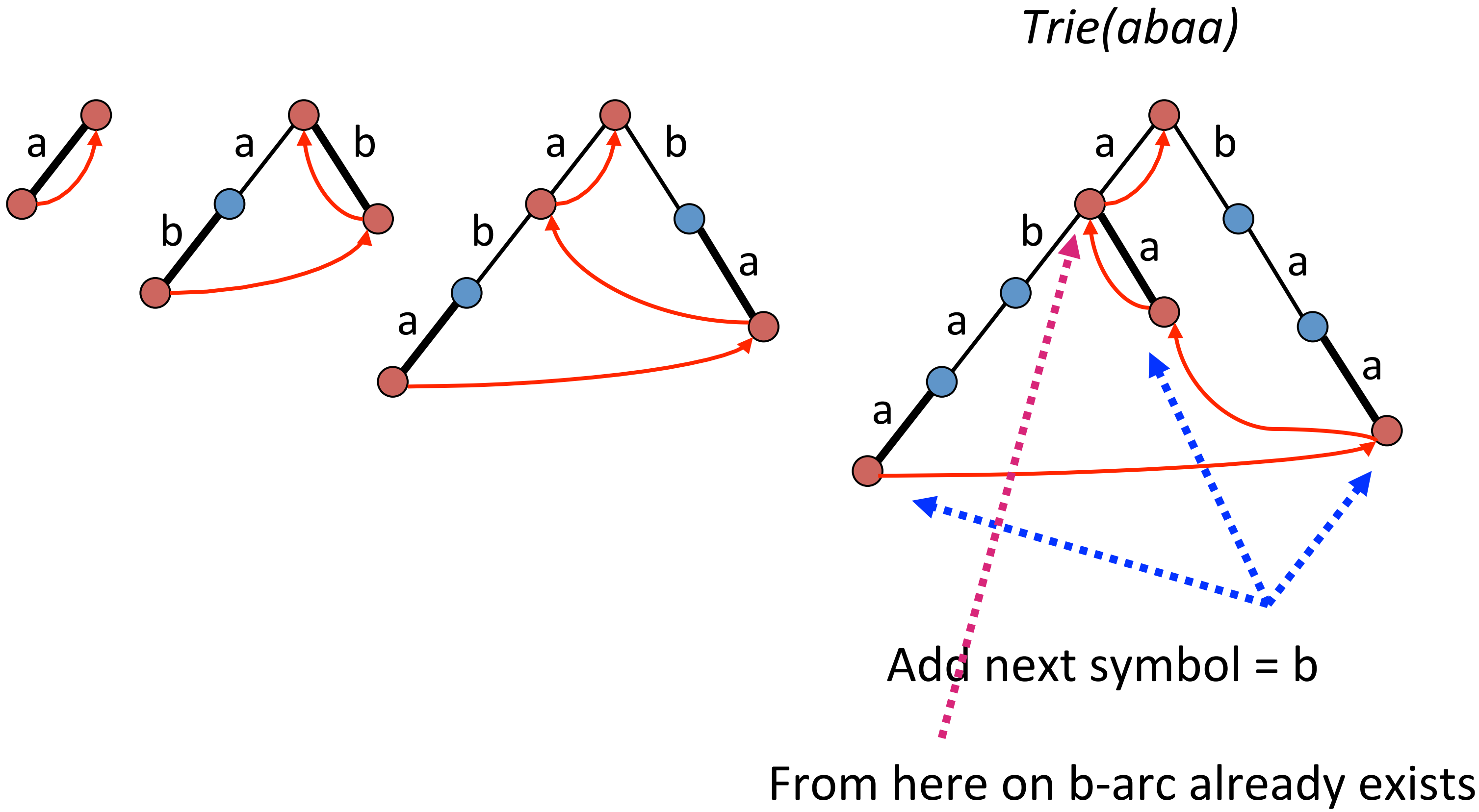
Trie(abaa)



Trie(abaab)

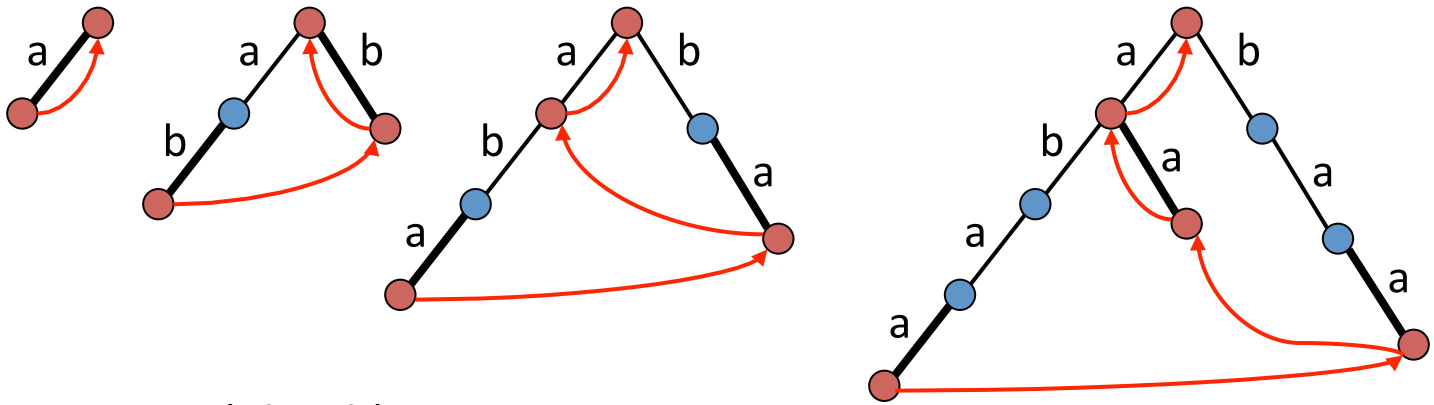


Trie(abaab)

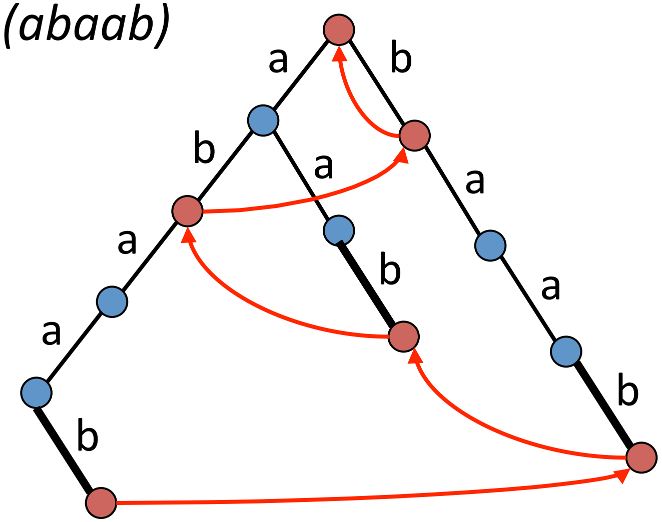


Trie(abaa)
a
a
b
b
a
b
b
a
a
a
b
b
a
a
a
a
a
a
Add next symbol = b
From here on b-arc already exists

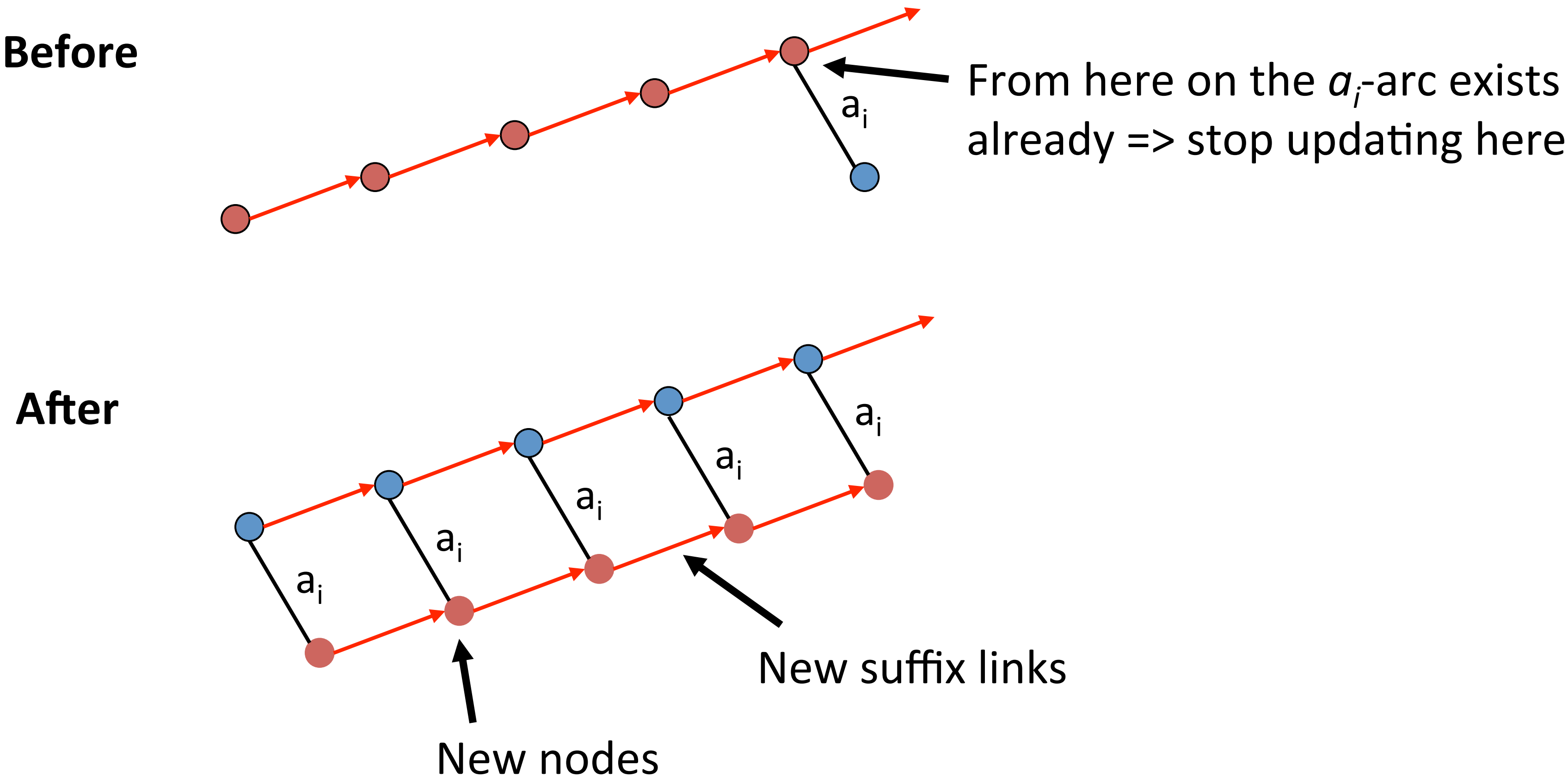
Trie(abaab)



Trie(abaab)



What happens in $Trie(P_i) \Rightarrow Trie(P_{i+1})$?



What happens in $Trie(P_i)$ => $Trie(P_{i+1})$?

- time: O(size of Trie(T))
- suffix links:

  slink(node($a\alpha$)) = node(α)

What can we do with it ?

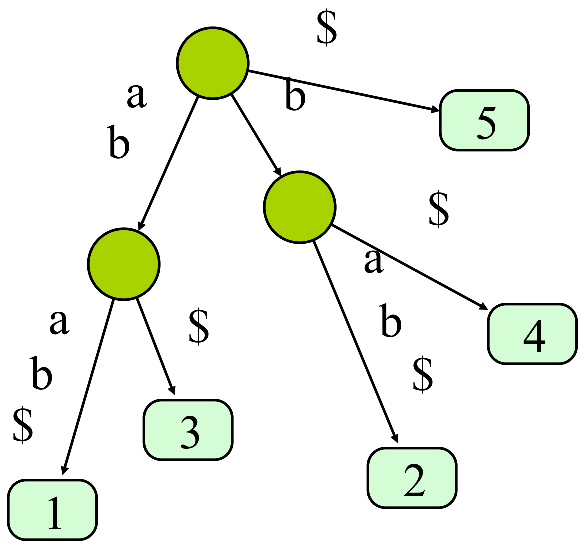
Exact string matching:

Given a Text T, |T| = n, preprocess it such that when a pattern P, |P|=m, arrives you can quickly decide when it occurs in T.

W e may also want to find all occurrences of P in T

Exact string matching

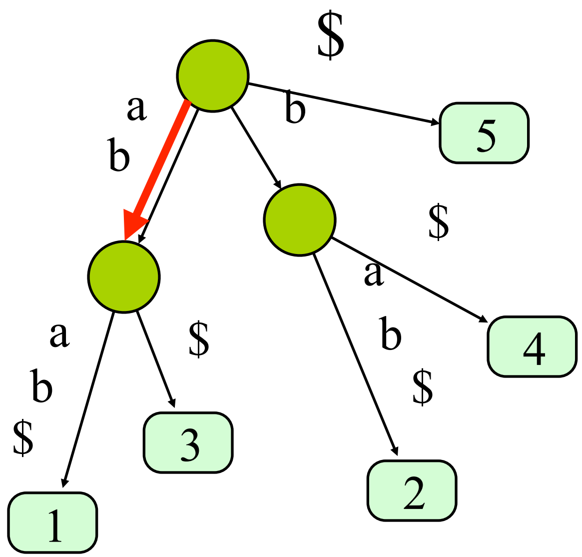
In preprocessing we just build a suffix tree in O(n) time



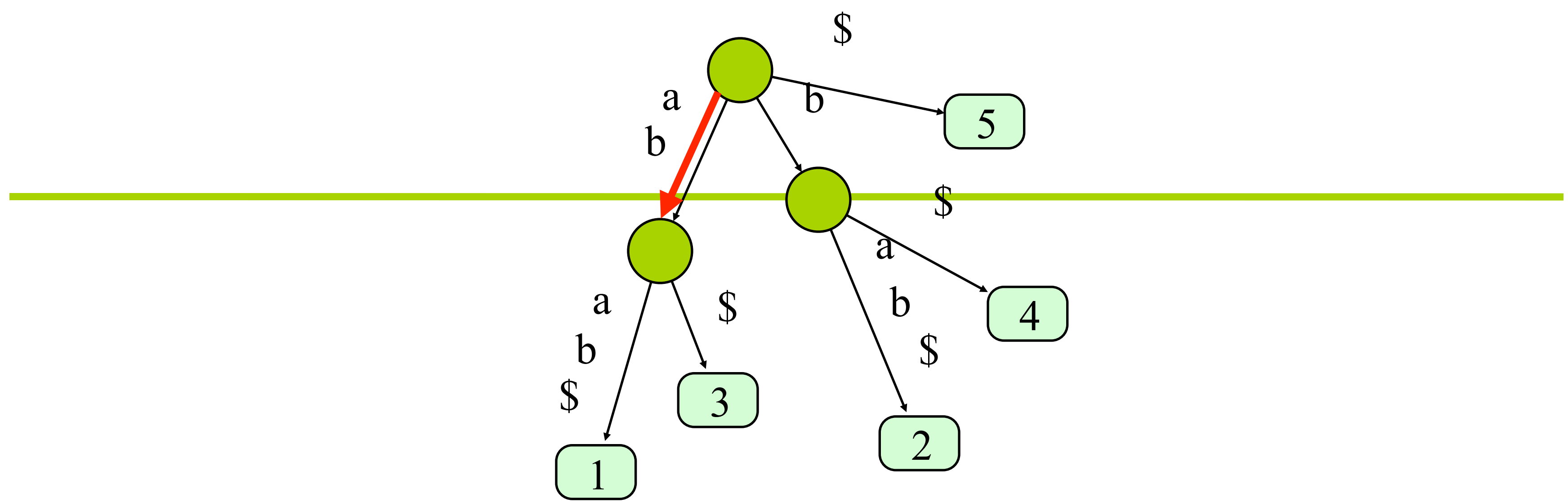
Given a pattern P = ab we traverse the tree according to the pattern.

Exact string matching

In preprocessing we just build a suffix tree in O(n) time



Given a pattern P = ab we traverse the tree according to the pattern.



If we did not get stuck traversing the pattern then the pattern occurs in the text.

Each leaf in the subtree below the node we reach corresponds to an occurrence.

By traversing this subtree we get all k occurrences in O(n+k) time

Generalized suffix tree

Given a set of strings S a generalized suffix tree of S is a compressed trie of all suffixes of $s \in S$

To make these suffixes prefix-free we add a special char, say $, at the end of s

To associate each suffix with a unique string in S add a different special char to each s

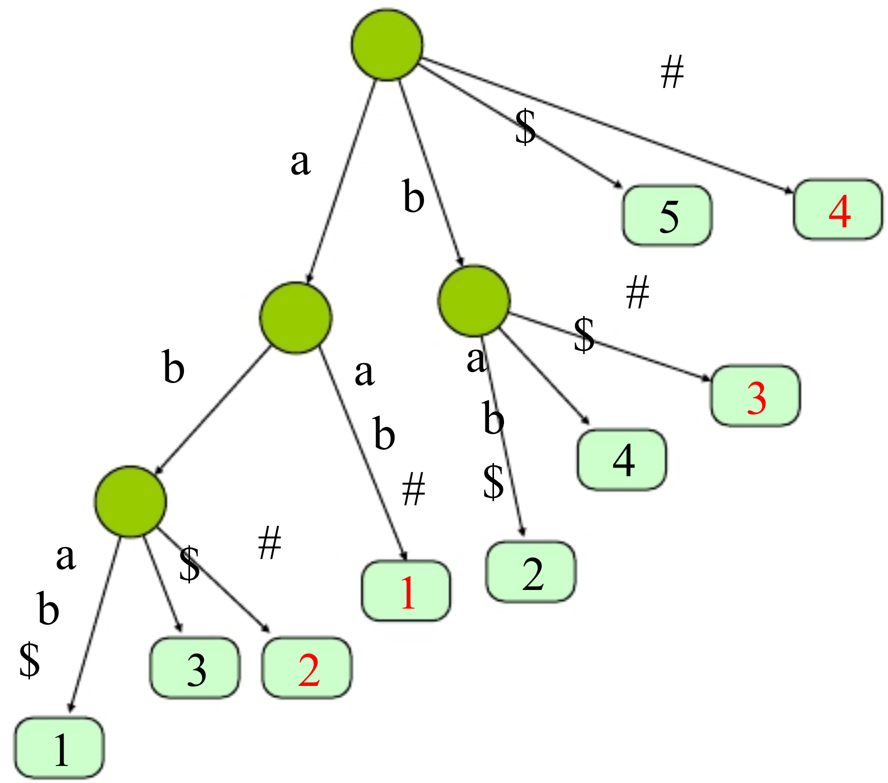
Generalized suffix tree (Example)

Let s_1=abab and s_2=aab here is a generalized suffix tree for s_1 and s_2

{

\$	#
b\$	b#
ab\$	ab#
bab\$	aab#
abab\$	

}



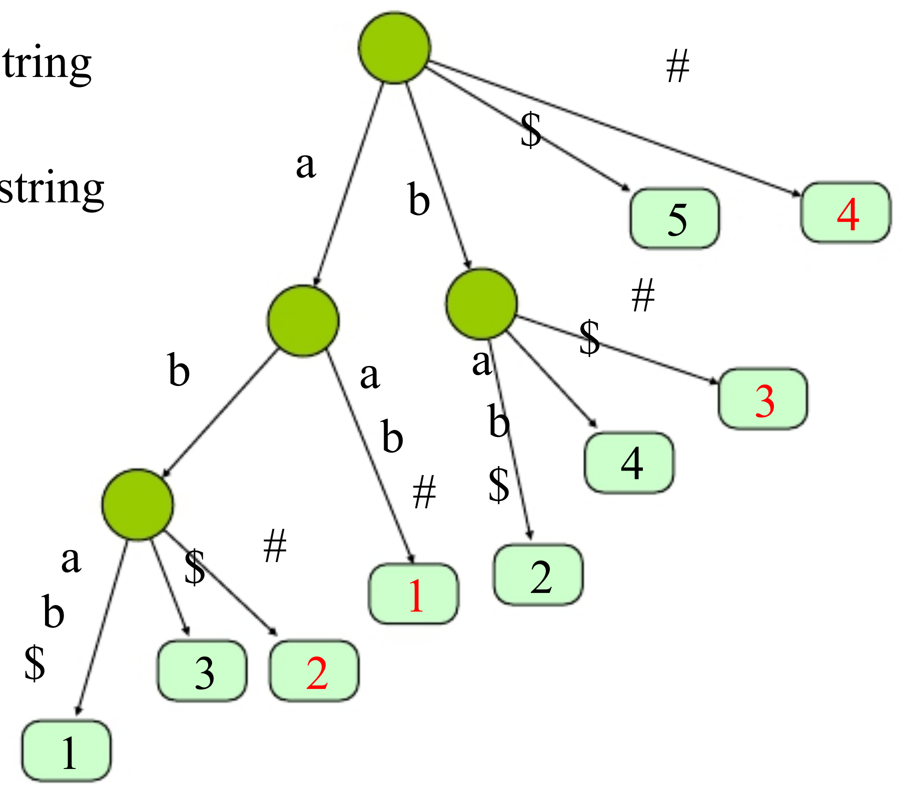
So what can we do with it ?

Matching a pattern against a database of strings

Longest common substring (of two strings)

Every node with a leaf descendant from string S1 and a leaf descendant from string S2 represents a maximal common substring and vice versa.

Find such node with largest "string depth"



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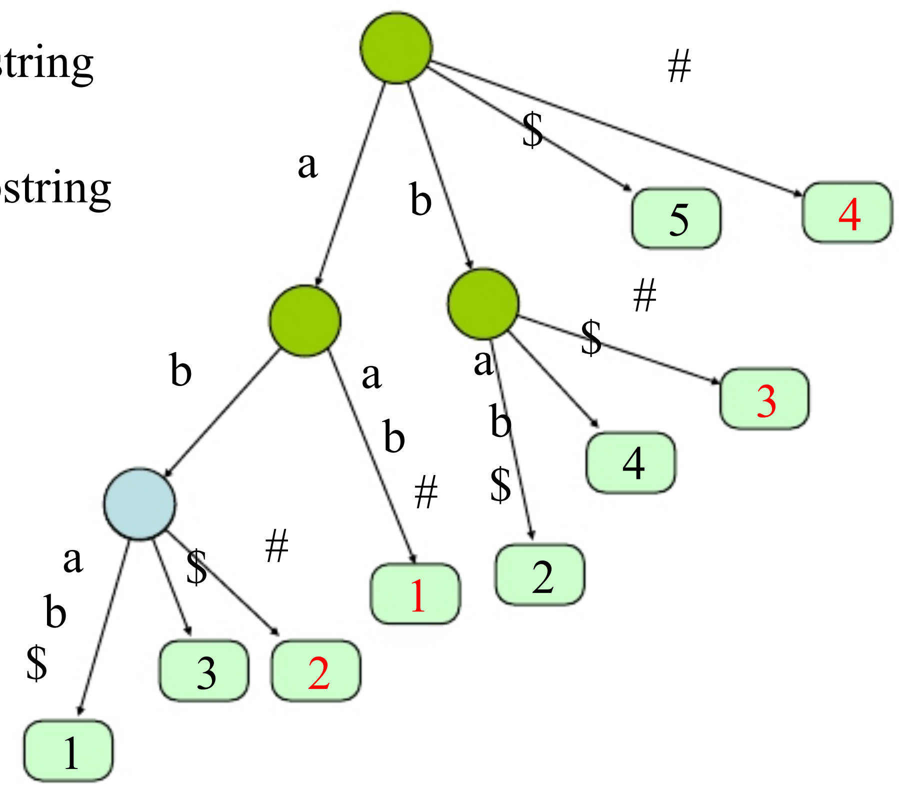
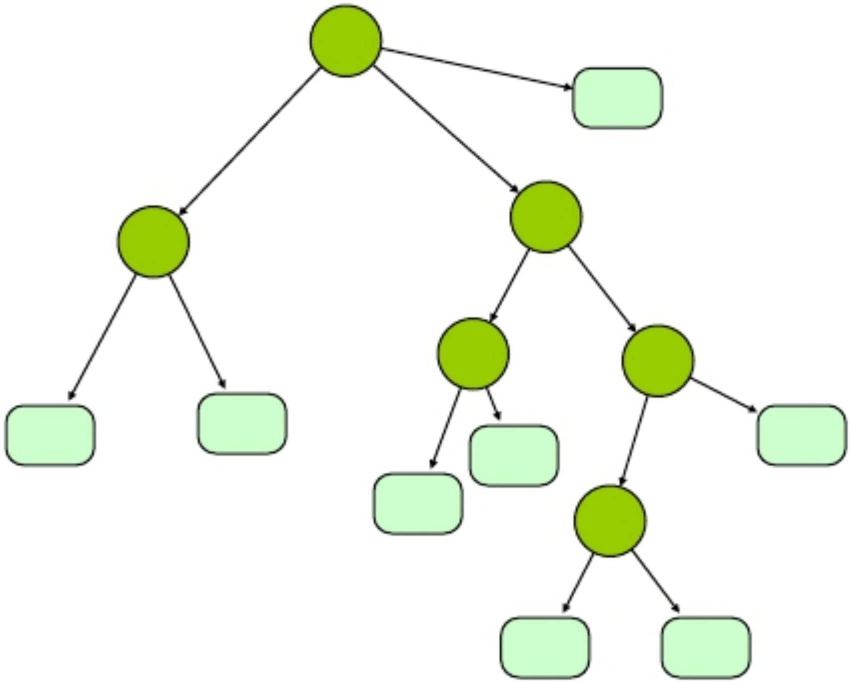


Lowest common ancestor

A lot more can be gained from the suffix tree if we preprocess it so that we can answer LCA queries on it



Lowest common ancestor

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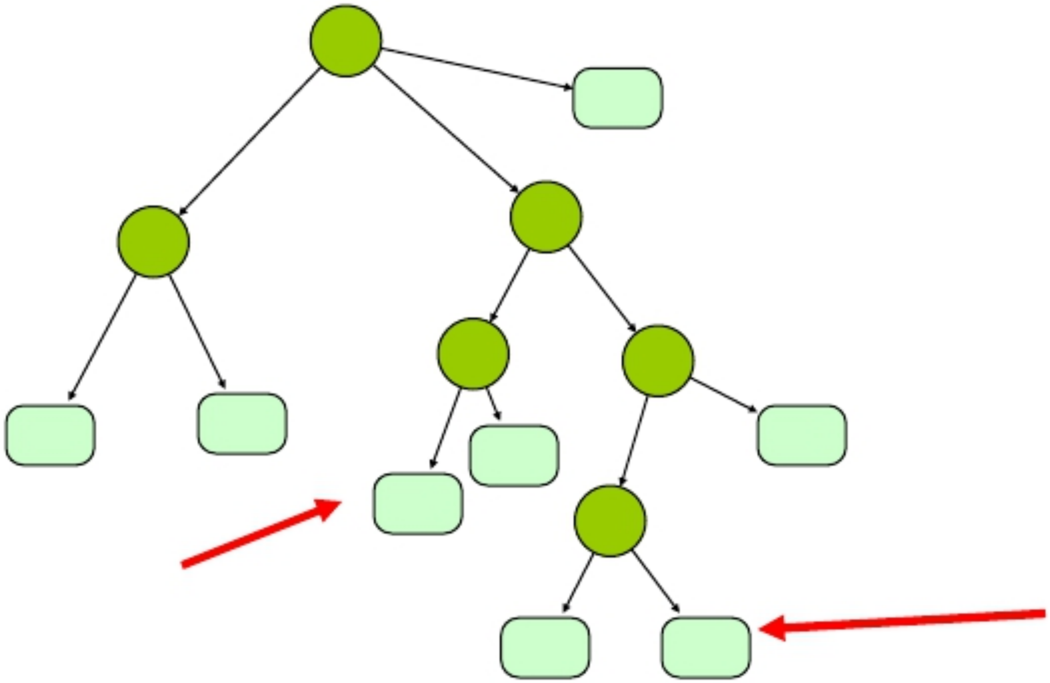

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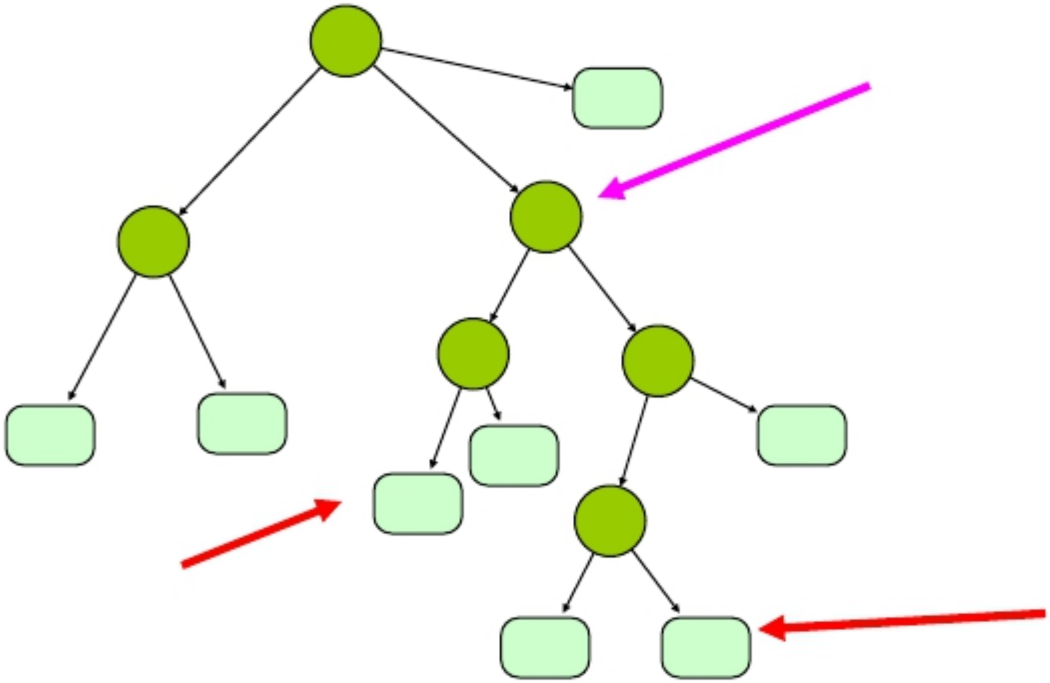
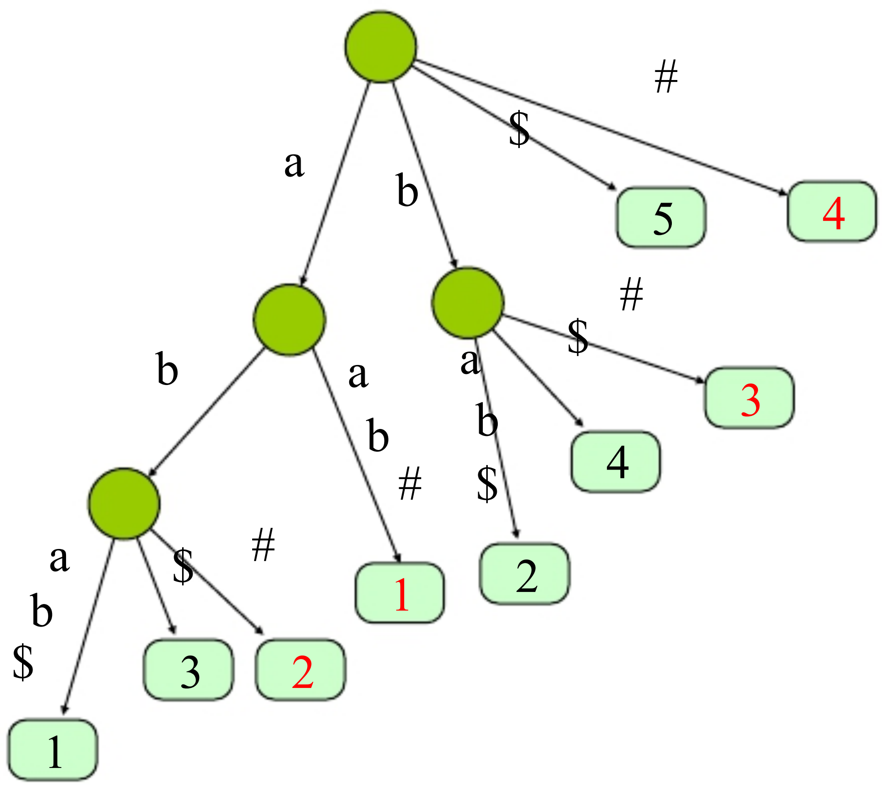


Why?

The LCA of two leaves represents the longest common prefix (LCP) of these 2 suffixes



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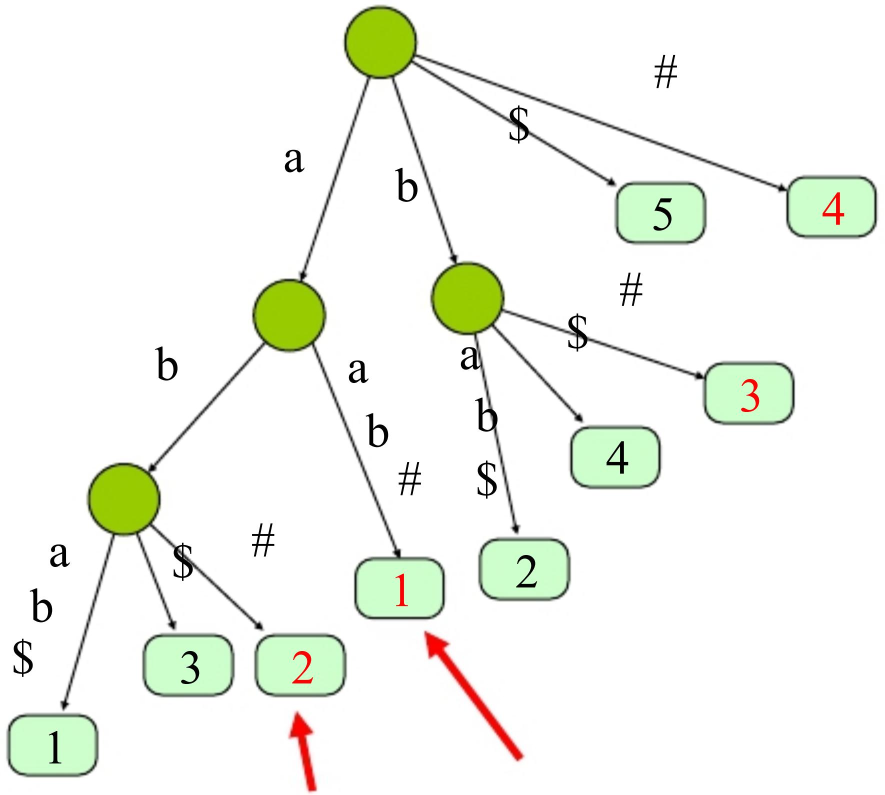
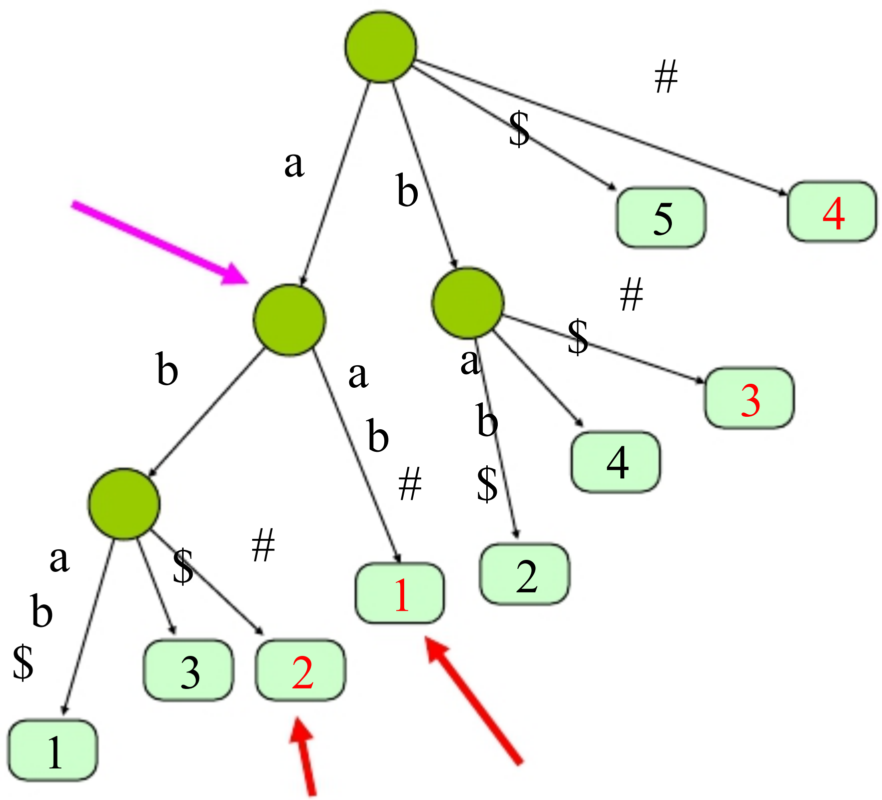


Why?

The LCA of two leaves represents the longest common prefix (LCP) of these 2 suffixes



Finding maximal palindromes

- A palindrome: caabaac, cbaabc
- Want to find all maximal palindromes in a string s

Let s = cbaaba

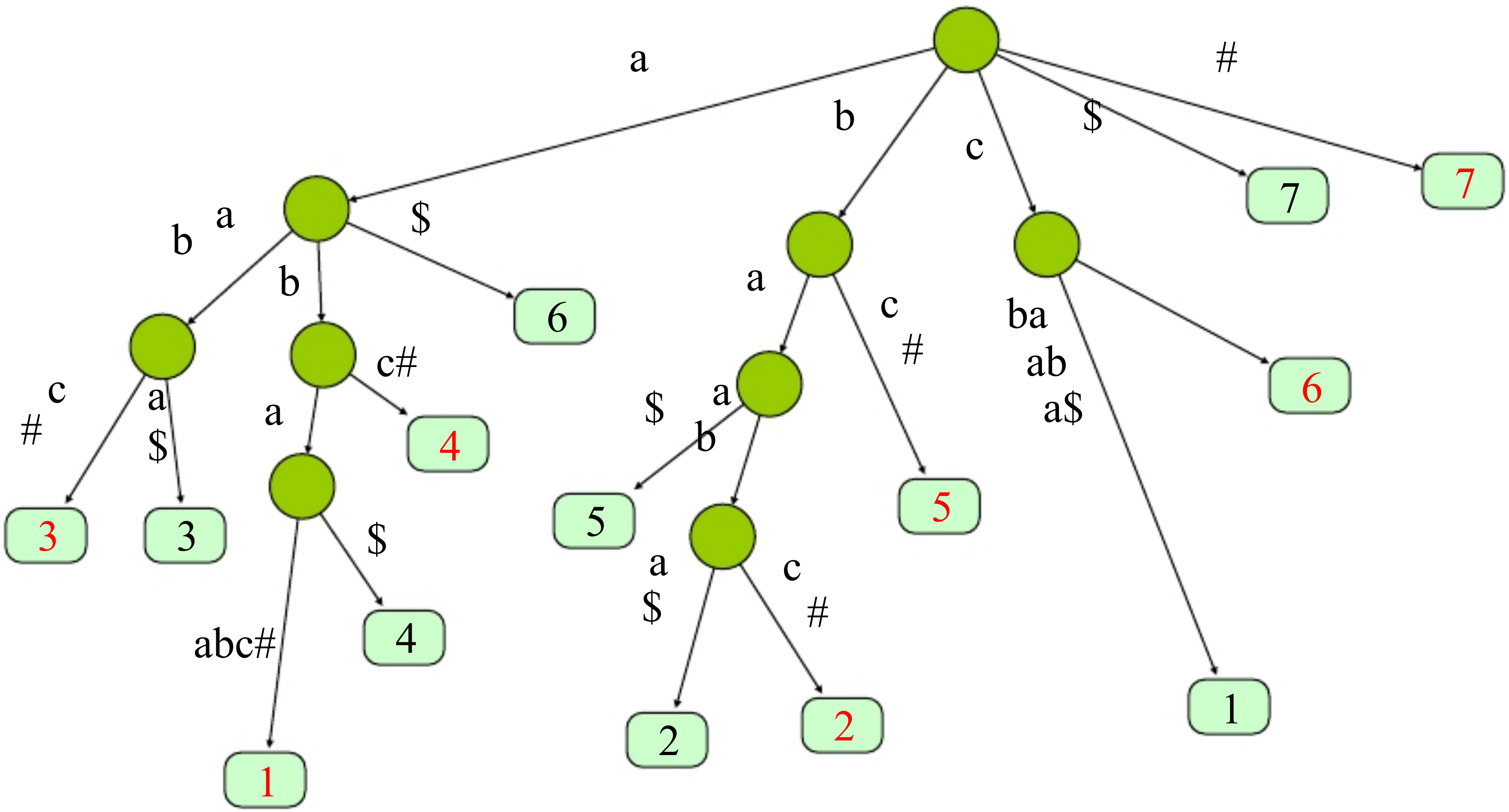
The maximal palindrome with center between i-1 and i is the LCP of the suffix at position i of **S** and the suffix at position m-i+1 of Sr

Maximal palindromes algorithm

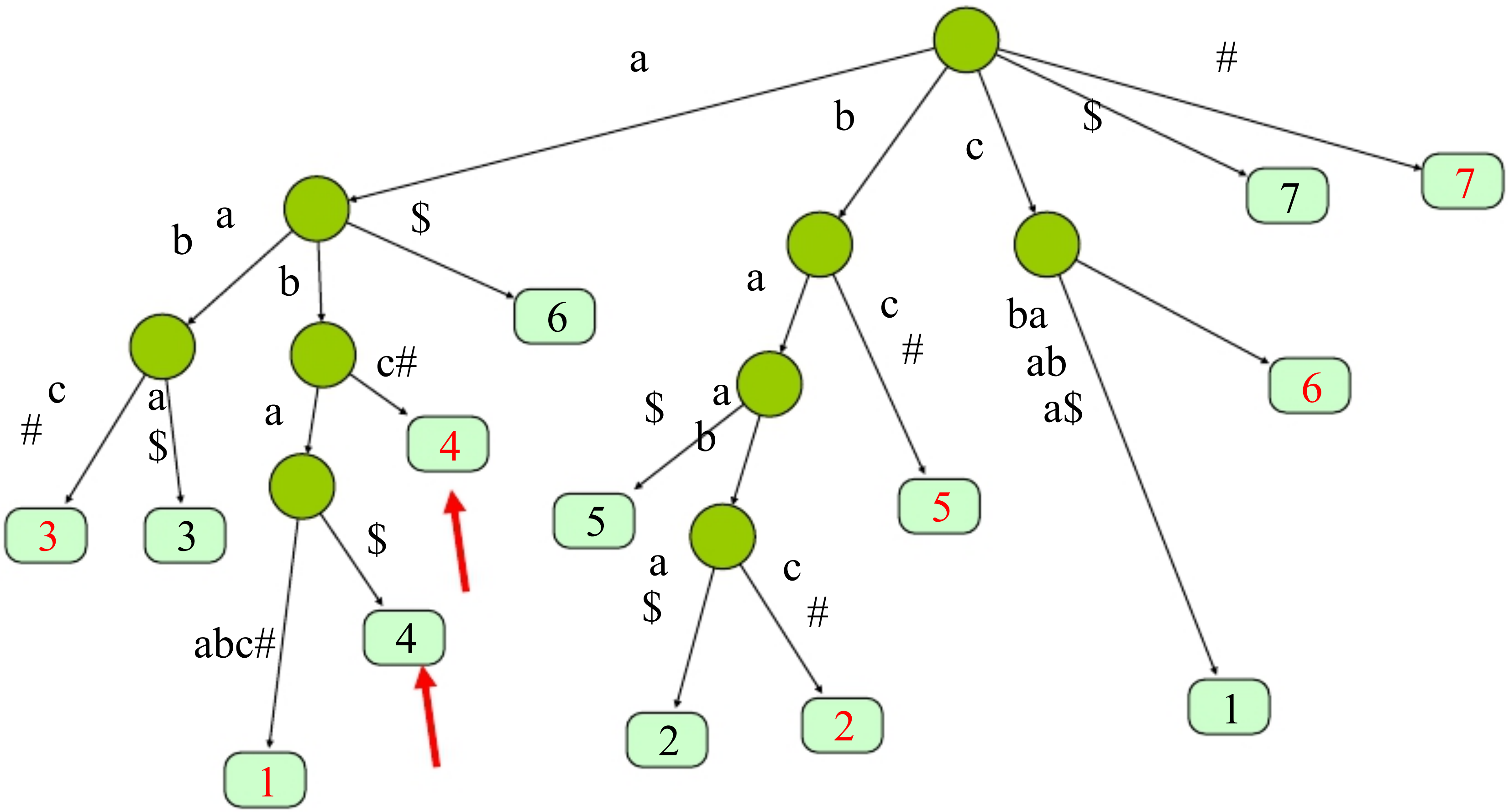
Prepare a generalized suffix tree for
s = cbaaba$ and s_r = abaabc#

For every i find the LCA of suffix i of s and suffix m-i+1 of s_r

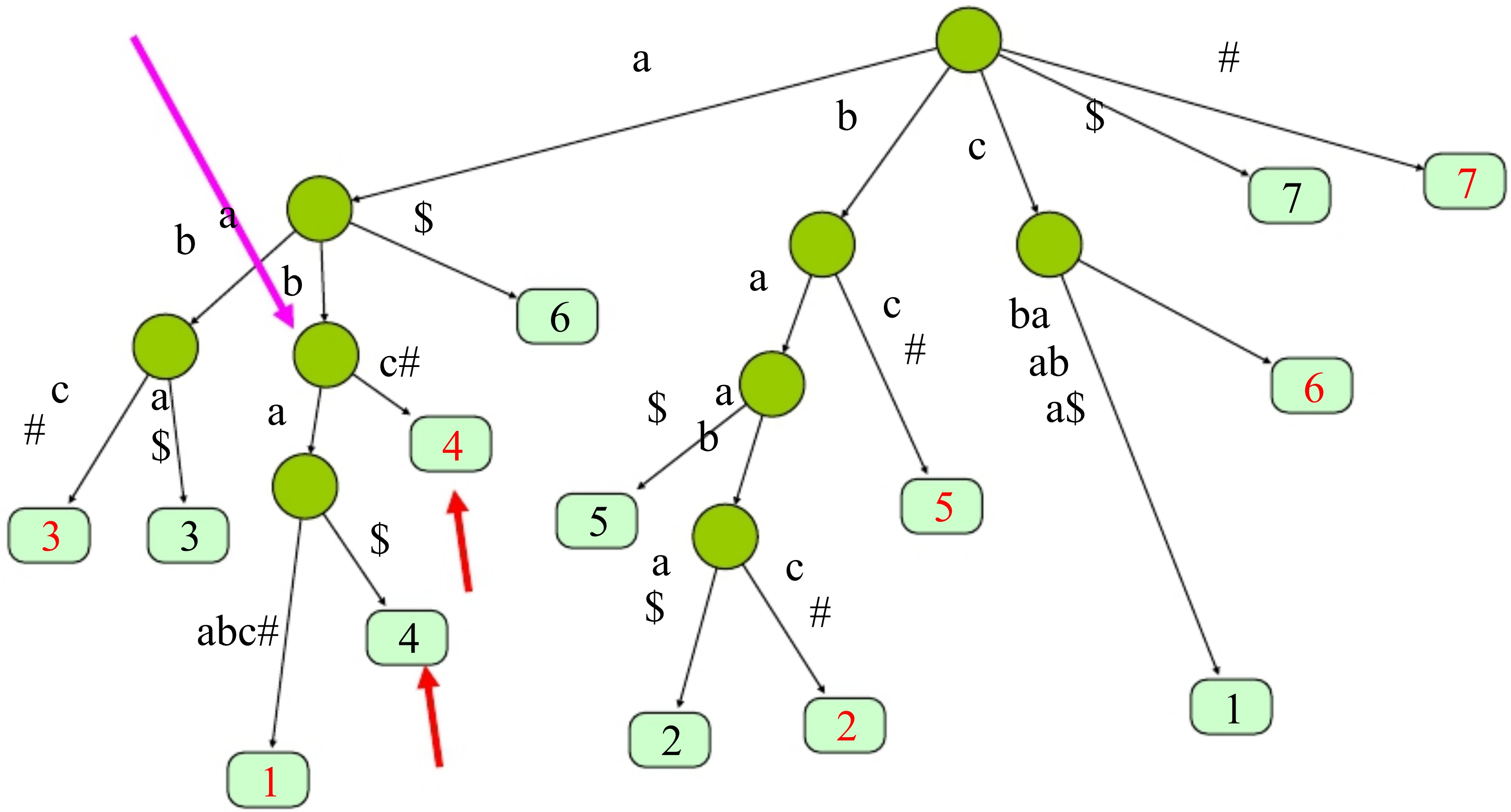
Let s = cbaaba$ then s_r = abaabc#



Let $s = \text{cbaaba\$}$ then $s_r = \text{abaabc\#}$



Let s = cbaaba$ then s_r = abaabc#



Analysis

O(n) time to identify all palindromes

Drawbacks

- Suffix trees consume a lot of space
- It is O(n) but the constant is quite big
- Notice that if we indeed want to traverse an edge in O(1) time then we need an array of ptrs. of size $|\Sigma|$ in each node

Suffix array

- We loose some of the functionality but we save space.

Let s = abab

Sort the suffixes lexicographically:

ab, abab, b, bab

The suffix array gives the indices of the suffixes in sorted order

3	1	4	2

How do we build it ?

- Build a suffix tree
- Traverse the tree in DFS, lexicographically picking edges outgoing from each node and fill the suffix array.

- O(n) time

How do we search for a pattern ?

- If P occurs in T then all its occurrences are consecutive in the suffix array.
- Do a binary search on the suffix array
- Takes O(mlogn) time

Example

Let S = mississippi

Let P = issa

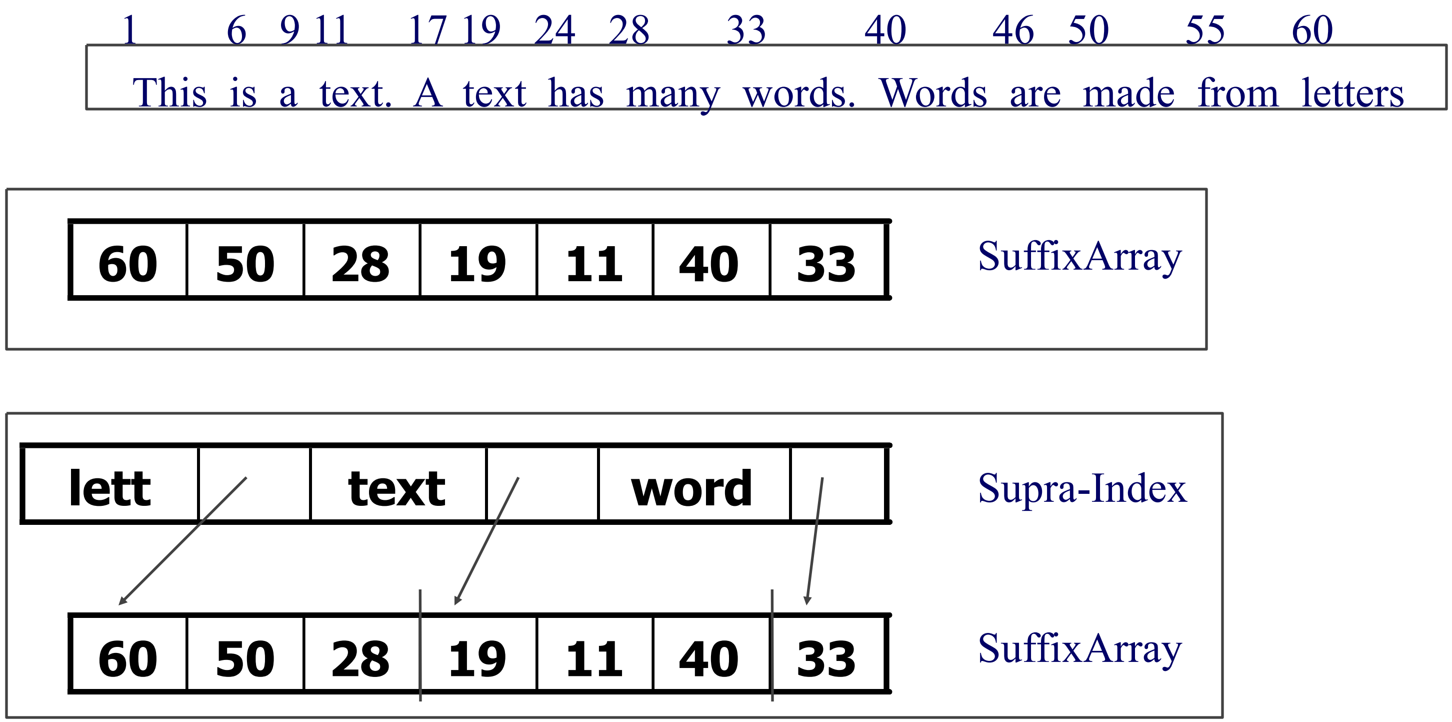
Pointer	Index	Suffix
L	11	i
	8	ippi
	5	issippi
	2	ississippi
	1	mississippi
M	10	pi
	9	ppi
	7	sippi
	4	sisippi
	6	ssippi
R	3	ssissippi

Supra index

- Structure
  - **Suffix arrays** are **space efficient** implementation of **suffix trees**.
  - Simply an array **containing all the pointers** to the text suffixes listed in lexicographical order.
  - **Supra-indices:**
    - If the suffix array is **large**, this binary search can perform **poorly** because of the number of random disk accesses.
    - Suffix arrays are designed to allow **binary searches** done by comparing the contents of each pointer.
    - To remedy this situation, the use of **supra-indices** over the suffix array has been proposed.

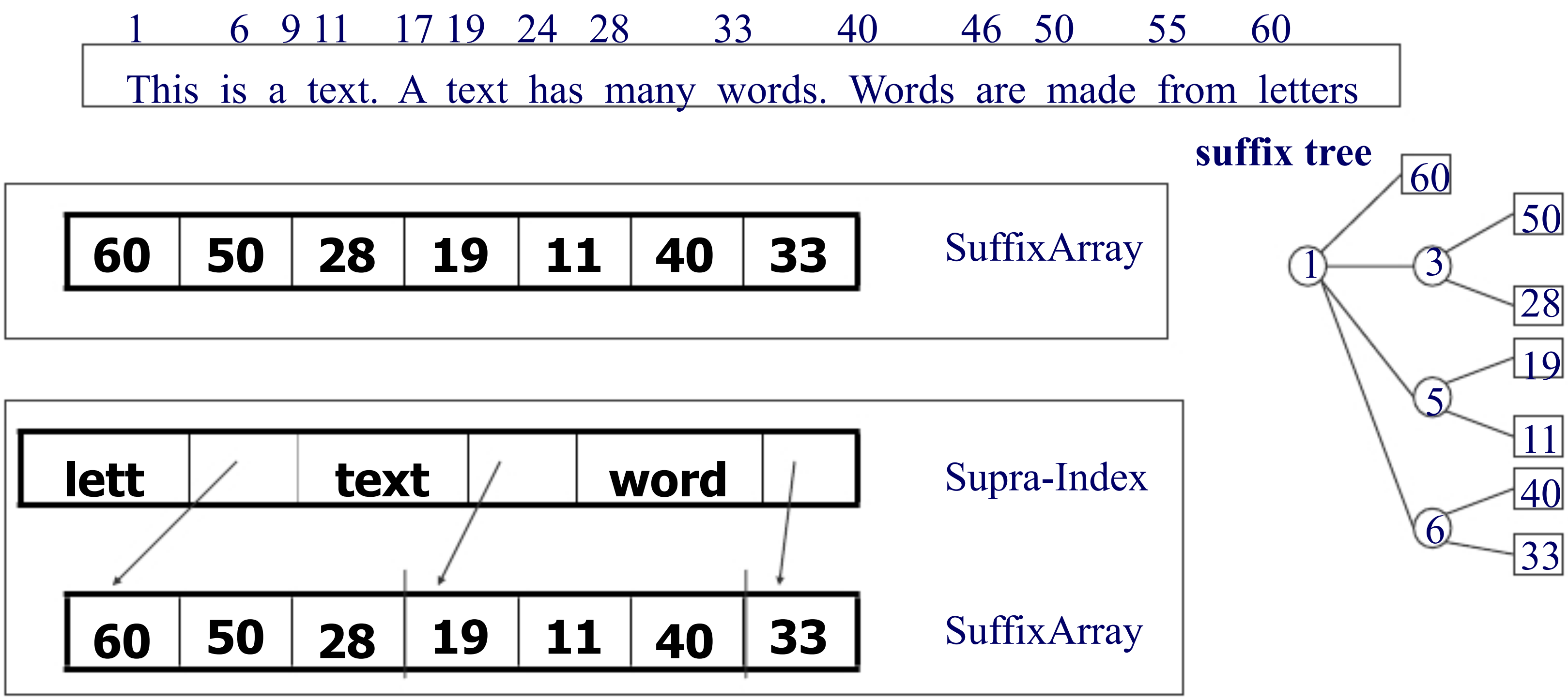
Supra index

- Example



Supra index

- Example



Tree(hattivatti)

hattivatti

attivatti

ttivatti

tivatti

ivatti

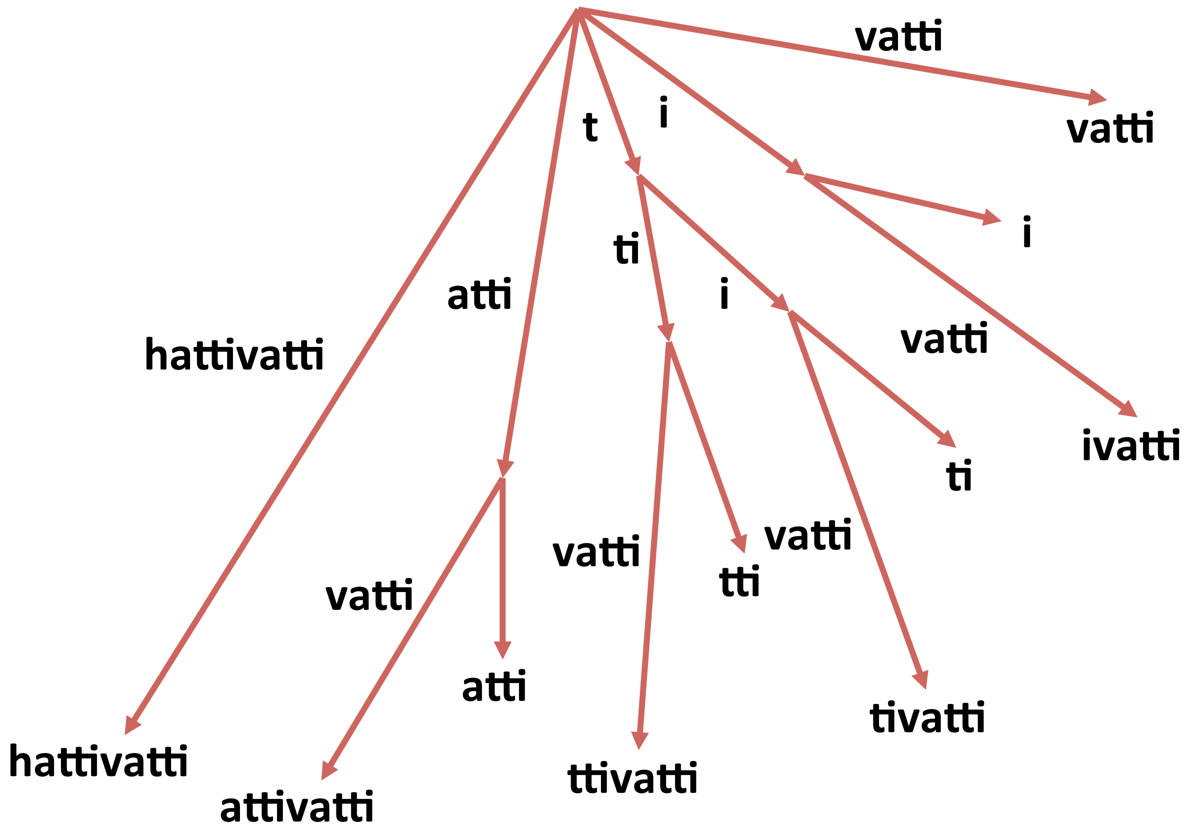
vatti

atti

tti

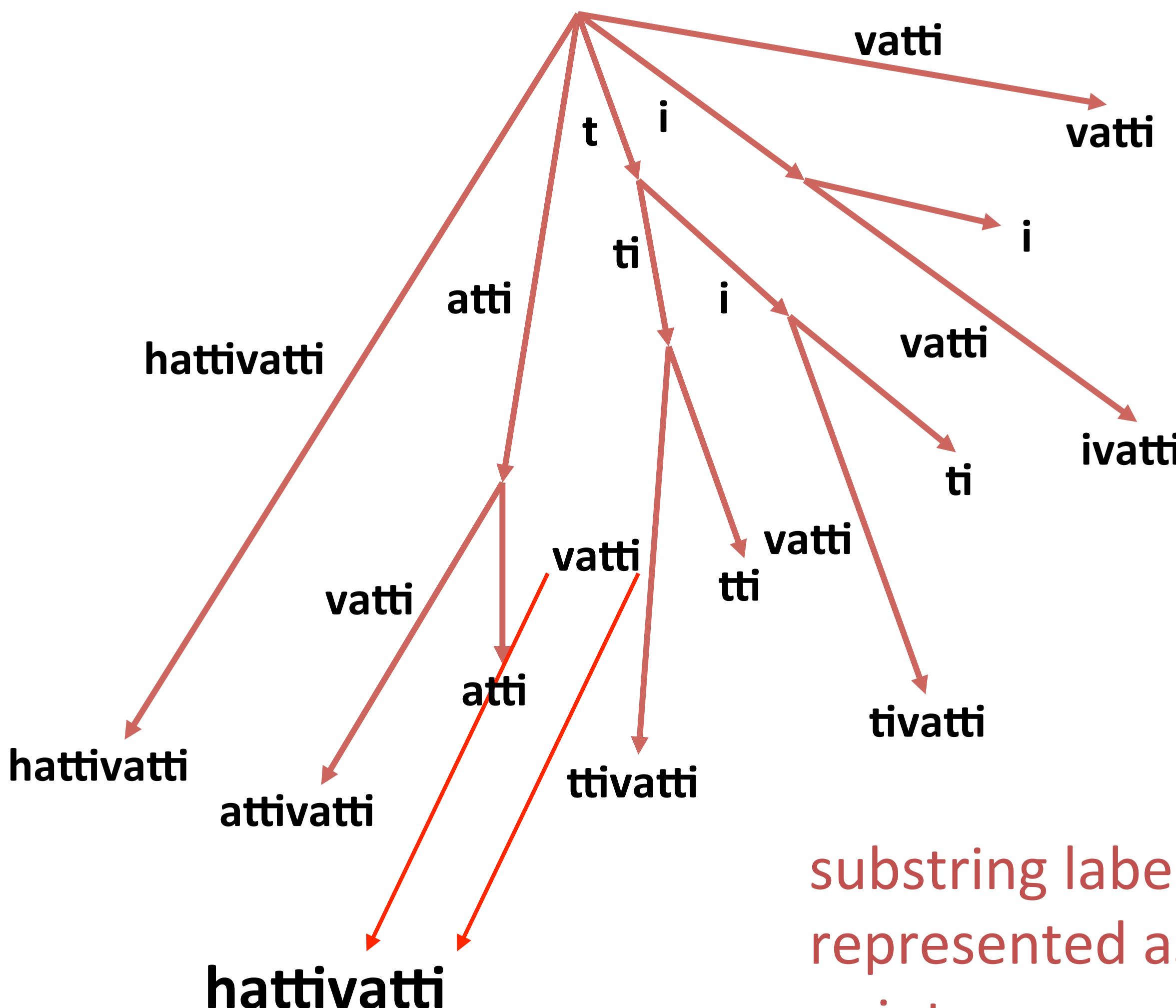
ti

i



Tree(hattivatti)

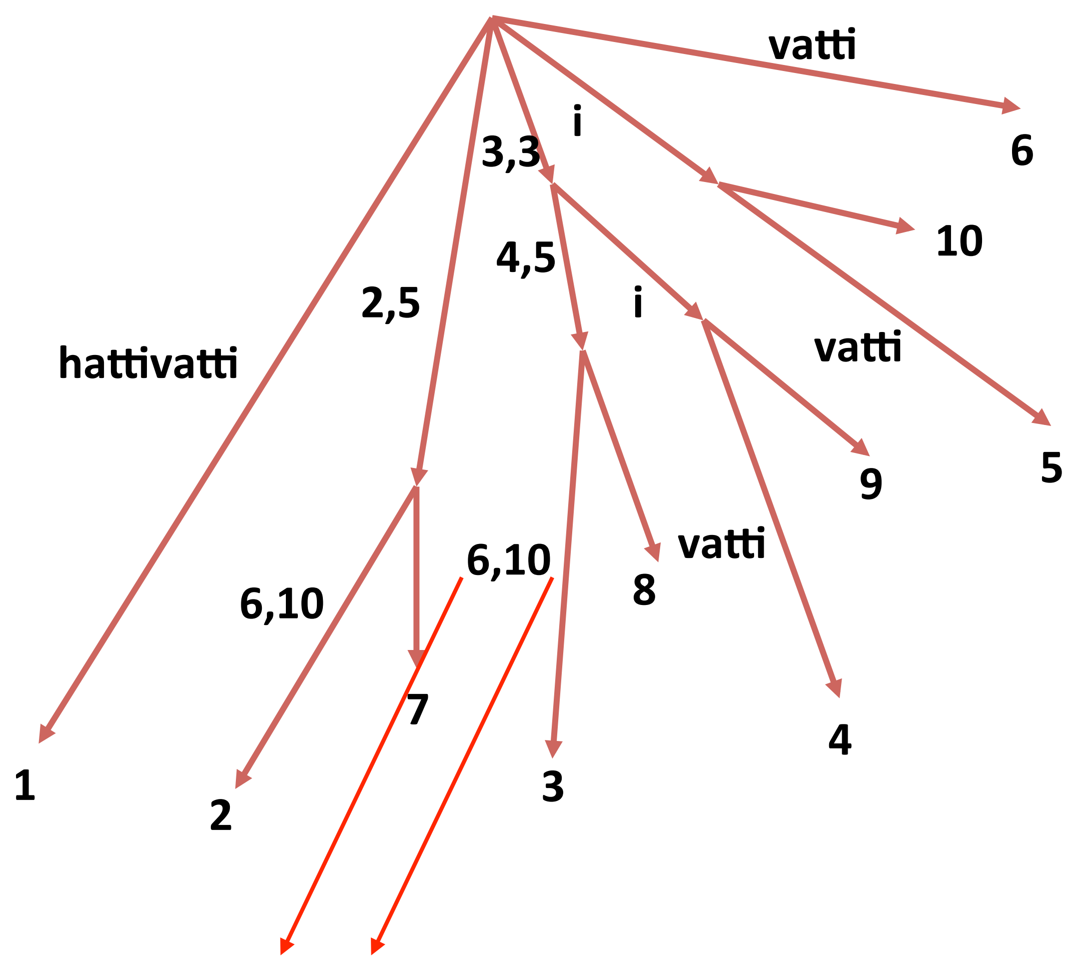
hattivatti
attivatti
ttivatti
tivatti
ivatti
vatti
atti
tti
ti
i



substring labels of edges represented as pairs of pointers

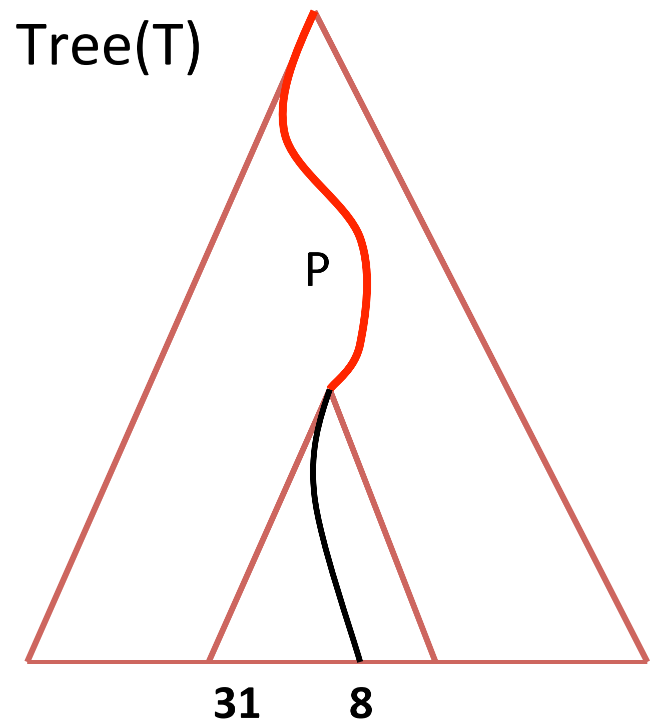
Tree(hattivatti)

hattivatti
attivatti
ttivatti
tivatti
ivatti
vatti
atti
tti
ti
i



hattivatti

Tree(T) is *full* text index



P occurs in T at locations 8, 31, …

P occurs in T ⇔ P is a prefix of some suffix of T ⇔ Path for P exists in Tree(T)

All occurrences of P in time O(|P| + #occ)

Find att from Tree(hattivatti)

hattivatti

attivatti

ttivatti

tivatti

ivatti

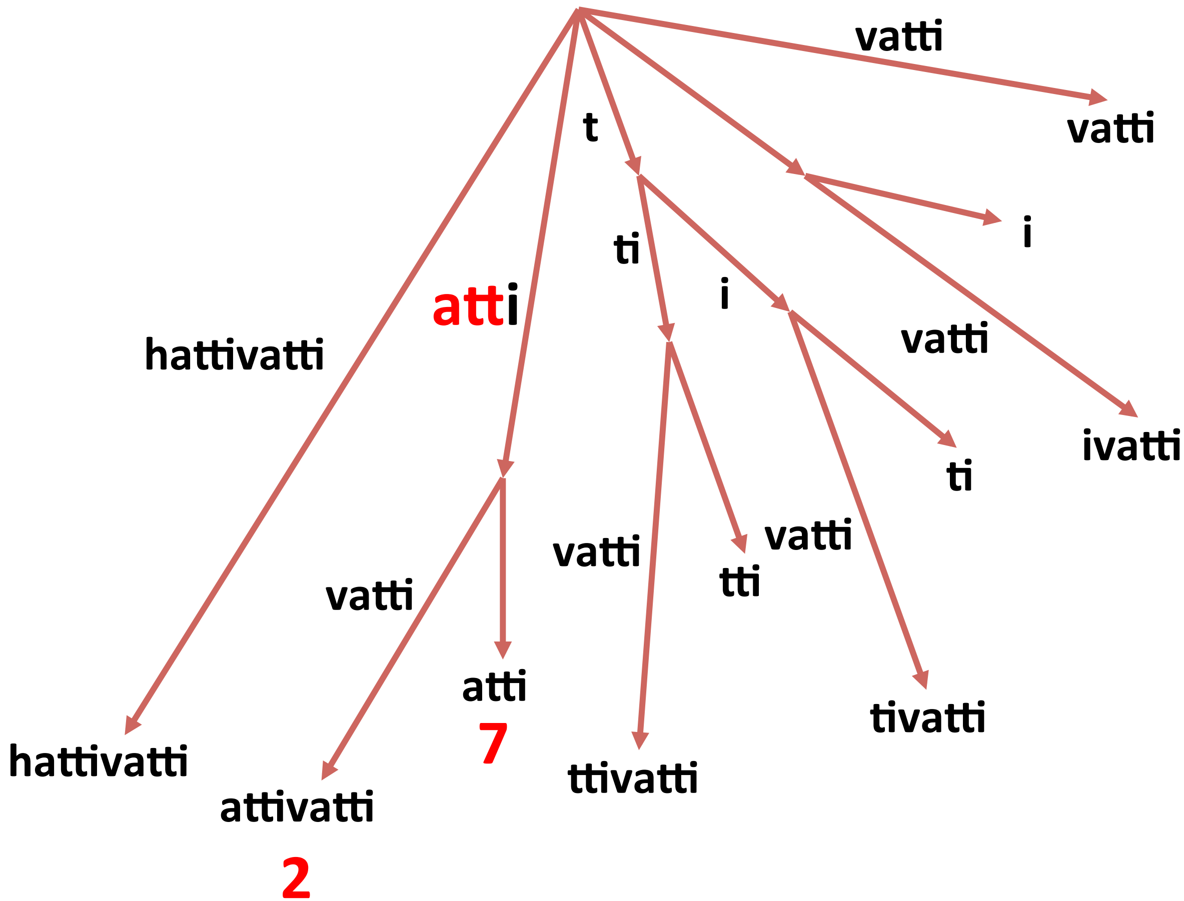
vatti

atti

tti

ti

i



Linear time construction of Tree(T)

hattivatti
attivatti
ttivatti
tivatti
ivatti
vatti
atti
tti
ti
i

Weiner (1973), 'algorithm of the year'

McCreight (1976)

'on-line' algorithm (Ukkonen 1992)