Analysis Of Binomial Heaps

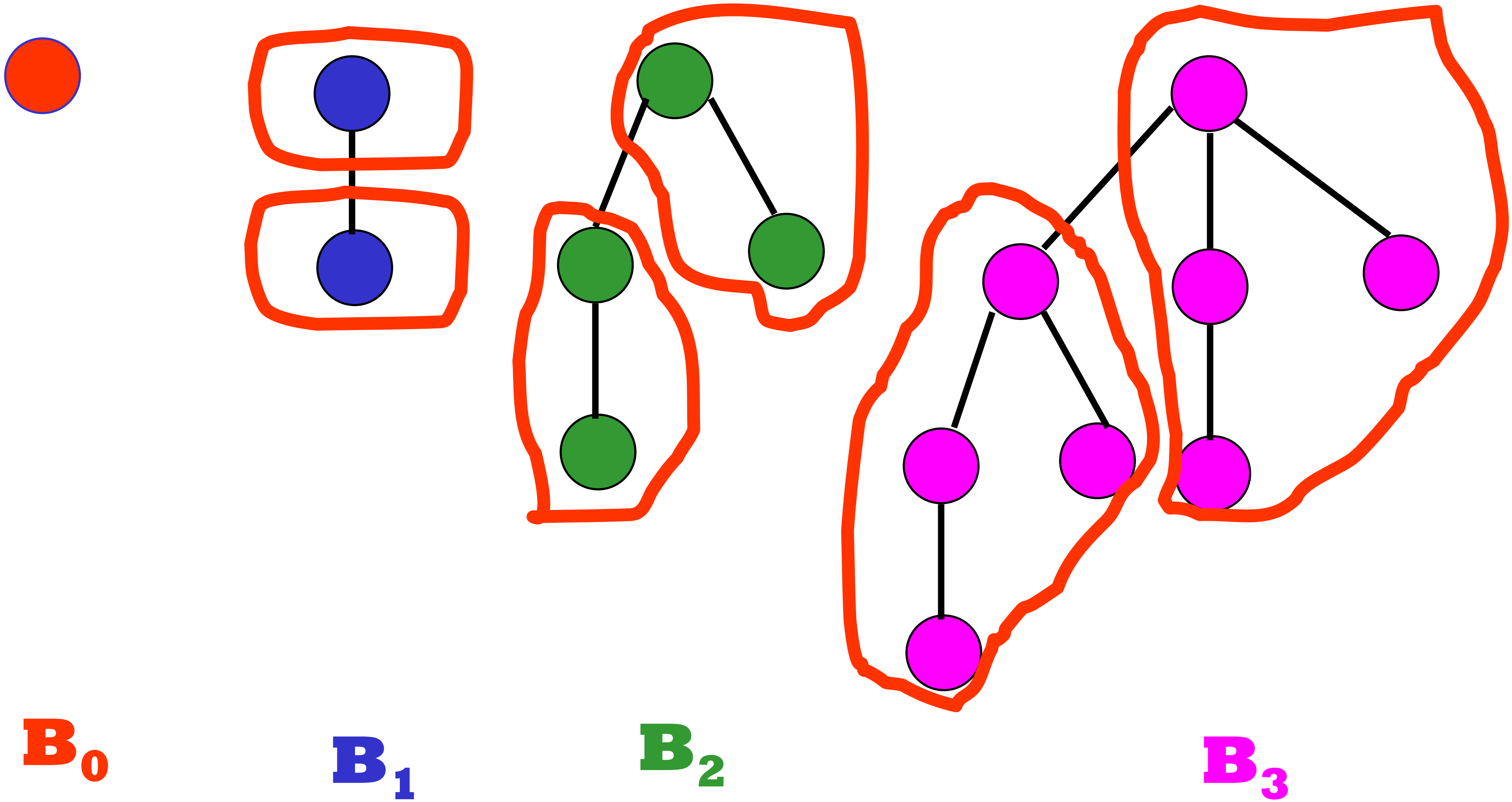
	Leftist trees	Binomial heaps	
		Actual	Amortized
Insert	O(log n)	O(1)	O(1)
Remove min (or max)	O(log n)	O(n)	O(log n)
Meld	O(log n)	O(1)	O(1)

Operations

- Insert
  - Add a new min tree to top-level circular list.
- Meld
  - Combine two circular lists.
- Remove min
  - Pairwise combine min trees whose roots have equal degree.
  - O(MaxDegree + s), where s is number of min trees following removal of min element but before pairwise combining.

Binomial Trees

- B_k , $k > 0$, is two B_{k-1}s.
- One of these is a subtree of the other.



All Trees In Binomial Heap Are Binomial Trees

- Initially, all trees in system are Binomial trees (actually, there are no trees initially).
- Assume true before an operation, show true after the operation.
- Insert creates a B_0.
- Meld does not create new trees.
- Remove Min
  - Reinserted subtrees are binomial trees.
  - Pairwise combine takes two trees of equal degree and makes one a subtree of the other.

Complexity of Remove Min

- Let n be the number of operations performed.
  - Number of inserts is at most n.
  - No binomial tree has more than n elements.
  - MaxDegree <= $\log_2 n$.
  - Complexity of remove min is $O(\log n + s) = O(n)$.

Aggregate Method

- Get a good bound on the cost of every sequence of operations and divide by the number of operations.
- Results in same amortized cost for each operation, regardless of operation type.
- Can’t use this method, because we want to show a different amortized cost for remove mins than for inserts and melds.

Aggregate Method – Alternative

- Get a good bound on the cost of every sequence of remove mins and divide by the number of remove mins.
- Consider the sequence insert, insert, …, insert, remove min.
  - The cost of the remove min is $O(n)$, where n is the number of operations in the sequence.
  - So, amortized cost of a remove min is $O(n/1) = O(n)$.

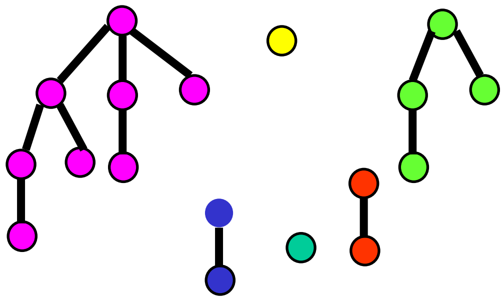
Accounting Method

- Guess the amortized cost.
  - Insert => 2.
  - Meld => 1.
  - Remove min => $3\log_2 n$.
- Show that $P(i) - P(0) >= 0$ for all i.

Potential Function

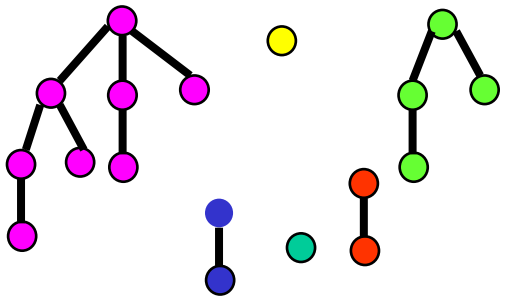
- $P(i) = amortizedCost(i) – actualCost(i) + P(i – 1)$
- $P(i) – P(0)$ is the amount by which the first i operations have been over charged.
- We shall use a credit scheme to keep track of (some of) the over charge.
- There will be 1 credit on each min tree.
- Initially, #trees = 0 and so total credits and $P(0) = 0$.
- Since number of trees cannot be <0, the total credits is always >= 0 and hence $P(i) >= 0$ for all i.

Insert



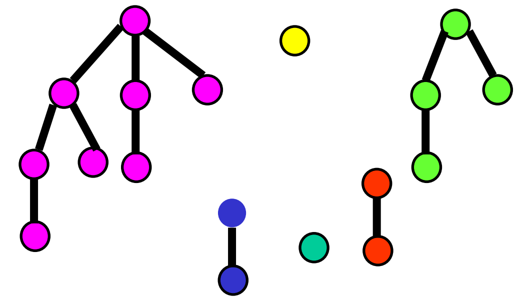
- Guessed amortized cost = 2.
- Use 1 unit to pay for the actual cost of the insert.
- Keep the remaining 1 unit as a credit.
- Keep this credit with the min tree that is created by the insert operation.
- Potential increases by 1, because there is an overcharge of 1.

Meld



- Guessed amortized cost = 1.
- Use 1 unit to pay for the actual cost of the meld.
- Potential is unchanged, because actual and amortized costs are the same.

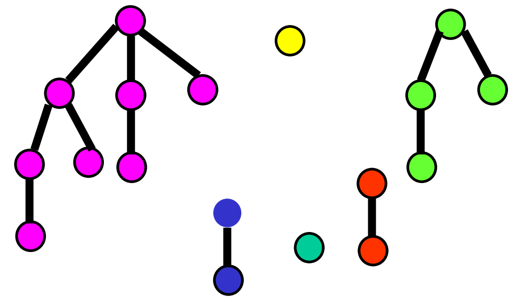
Remove Min



- Let MinTrees be the set of min trees in the binomial heap just before remove min.
- Let u be the degree of min tree whose root is removed.
- Let s be the number of min trees in binomial heap just before pairwise combining.
  - s = #MinTrees + u – 1
- Actual cost of remove min is <= MaxDegree + s

  $\leq 2\log_2 n - 1 + \#MinTrees$.

Remove Min



- Guessed amortized cost $= 3\log_2 n$.
- Actual cost $<= 2\log_2 n - 1 + \#MinTrees$.
- Allocation of amortized cost.
  - Use up to $2\log_2 n - 1$ to pay part of actual cost.
  - Keep some or all of the remaining amortized cost as a credit.
  - Put 1 unit of credit on each of the at most $\log_2 n + 1$ min trees left behind by the remove min operation.
  - Discard the remainder (if any).

Paying Actual Cost Of A Remove Min

- Actual cost $<= 2\log_2 n - 1 + \#MinTrees$
- How is it paid for?
  - $2\log_2 n - 1$ comes from amortized cost of this remove min operation.
  - #MinTrees comes from the min trees themselves, at the rate of 1 unit per min tree, using up their credits.
  - Potential may increase or decrease but remains nonnegative as each remaining tree has a credit.

Potential Method

- Guess a suitable potential function for which $P(i) - P(0) >= 0$ for all i.
- Derive amortized cost of ith operation using
  $\Delta P = P(i) - P(i - 1)$
  $\qquad = \text{amortized cost} - \text{actual cost}$
- amortized cost = actual cost + ΔP

Potential Function

- $P(i) = \Sigma \#MinTrees(j)$
  - #MinTrees(j) is #MinTrees for binomial heap j.
  - When binomial heaps A and B are melded, A and B are no longer included in the sum.
- $P(0) = 0$
- $P(i) >= 0$ for all i.
- ith operation is an insert.
  - Actual cost of insert = 1
  - $\Delta P = P(i) - P(i-1) = 1$
  - Amortized cost of insert = actual cost + ΔP
    = 2

ith Operation Is A Meld

- Actual cost of meld = 1
- $P(i) = \Sigma \#MinTrees(j)$
- $\Delta P = P(i) - P(i-1) = 0$
- Amortized cost of meld = actual cost + ΔP

  = 1

ith Operation Is A Remove Min

- old => value just before the remove min
- new => value just after the remove min.
- $\#MinTrees^{old}(j)$ => value of #MinTrees in jth binomial heap just before this remove min.
- Assume remove min is done in kth binomial heap.

ith Operation Is A Remove Min

- Actual cost of remove min from binomial heap k

  $<= 2\log_2 n - 1 + \#MinTrees^{old}(k)$

- $\Delta P = P(i) - P(i - 1)$

  $= \Sigma[\#MinTrees^{new}(j) - \#MinTrees^{old}(j)]$

  $= \#MinTrees^{new}(k) - \#MinTrees^{old}(k)$.

- Amortized cost of remove min = actual cost + ΔP

  $<= 2\log_2 n - 1 + \#MinTrees^{new}(k)$

  $<= 3\log_2 n$.