Web Data Compression and Search

Fast BWT Construction

Space Efficient Linear Time Construction of Suffix Arrays

A good paper by Pang Ko and Srinivas Aluru

J. Discrete Algorithms 3(2-4): 143-156(2005)

Suffix Array

- Sorted order of suffixes of a string *T*.
- Represented by the starting position of the suffix.

Text	M	I	S	S	I	S	S	I	P	P	I	$
Index	1	2	3	4	5	6	7	8	9	10	11	12
Suffix Array	12	11	8	5	2	1	10	9	7	4	6	3

Notation

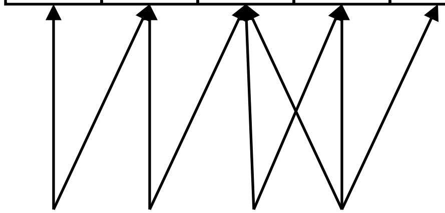
- String $T = t_1 \ldots t_n$.
- Over the alphabet $\Sigma = \{1 \ldots n\}$.
- t_n = ‘$’, ‘$’ is a unique character.
- $T_i = t_i \ldots t_n$, denotes the i-th suffix of T.
- For strings α and β, $\alpha < \beta$ denotes α is lexicographically smaller than β.

Overview

- Divide all suffixes of *T* into two types.
  - Type S suffixes = $\{T_i \mid T_i < T_{i+1}\}$
  - Type L suffixes = $\{T_j \mid T_j > T_{j+1}\}$
  - The last suffix is both type *S* and *L*.
- Sort all suffixes of one of the types.
- Obtain lexicographical order of all suffixes from the sorted ones.

Identify Suffix Types

Type	L	S	L	L	S	L	L	S	L	L	L	L/S
Text	M	I	S	S	I	S	S	I	P	P	I	$



M > I

I < S

S = S, so check next character

S > I

⇒ $T_1 > T_2$

⇒ $T_2 < T_3$

⇒ $T_3 > T_4 > T_5$

⇒ T_1 is type L

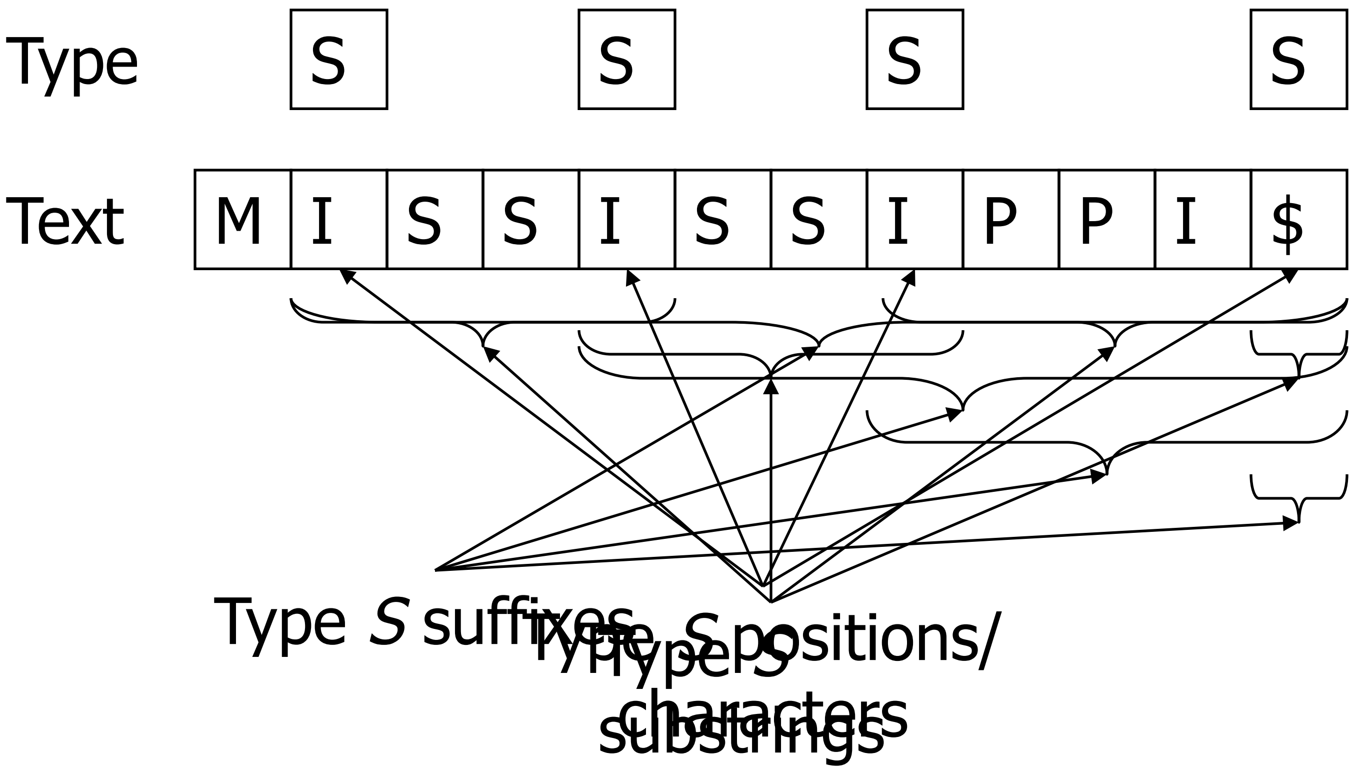
⇒ T_2 is type S

⇒ T_3 and T_4 are type L

The type of each suffix in T can be determined in one scan of the string.

In the suffix array of T, among all suffixes that start with the same character, the type S suffixes appear after the type L suffixes.

Notation

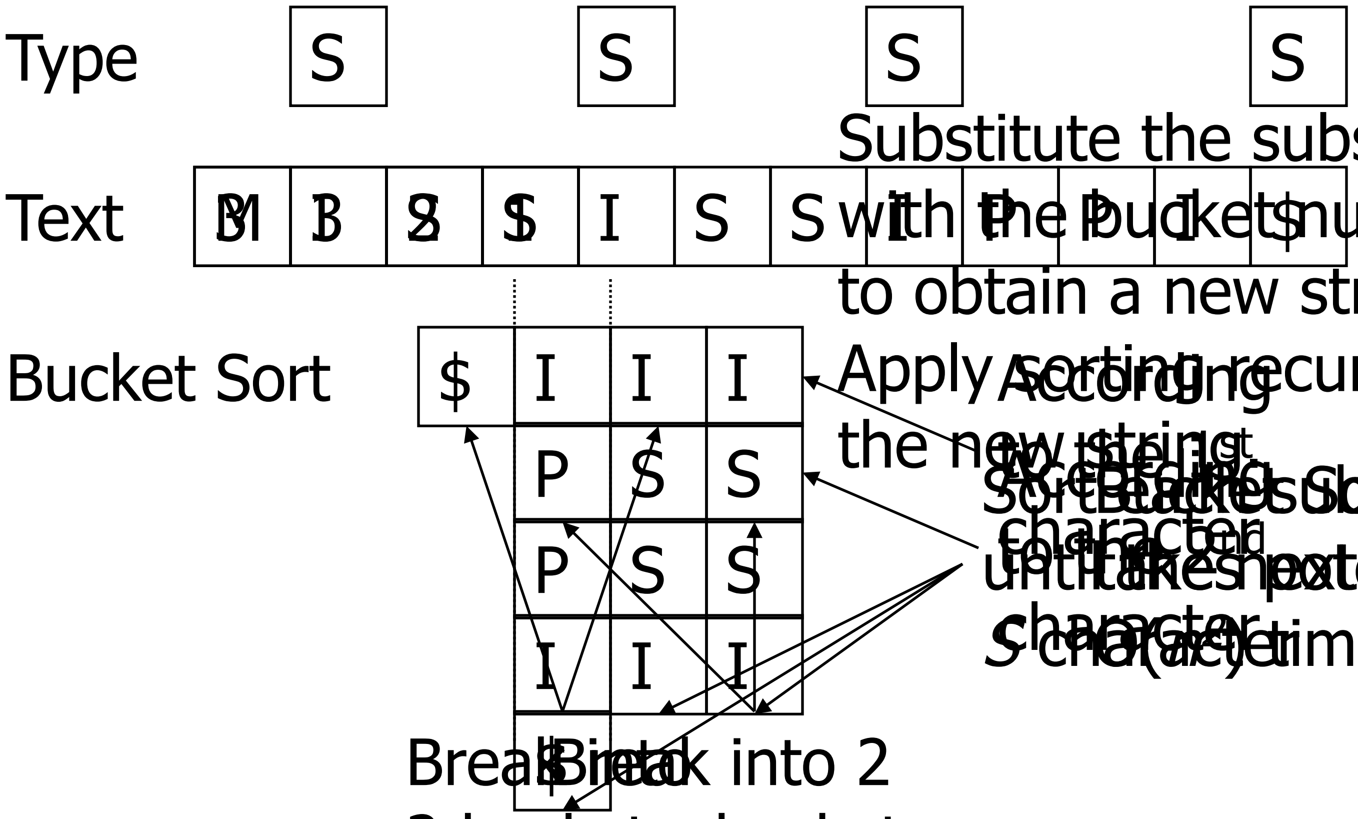


Type		S			S			S				S
Text	M	I	S	S	I	S	S	I	P	P	I	$

Sorting Type *S* Suffixes

- Sort all type *S* substrings.
- Replace each type *S* substrings by its bucket number.
- New string is the sequence of bucket numbers.
- Sorting all type *S* suffixes = Sorting all suffixes of the new string.

Sorting Type *S* Substrings



Substitute the substrings with the bucket numbers to obtain a new string. Apply sorting recursively to the new string.

According to the 1st character

According to the 2nd character

Sort each substring until the next type *S* character

Bucket Sort takes potentially O(*n*) time

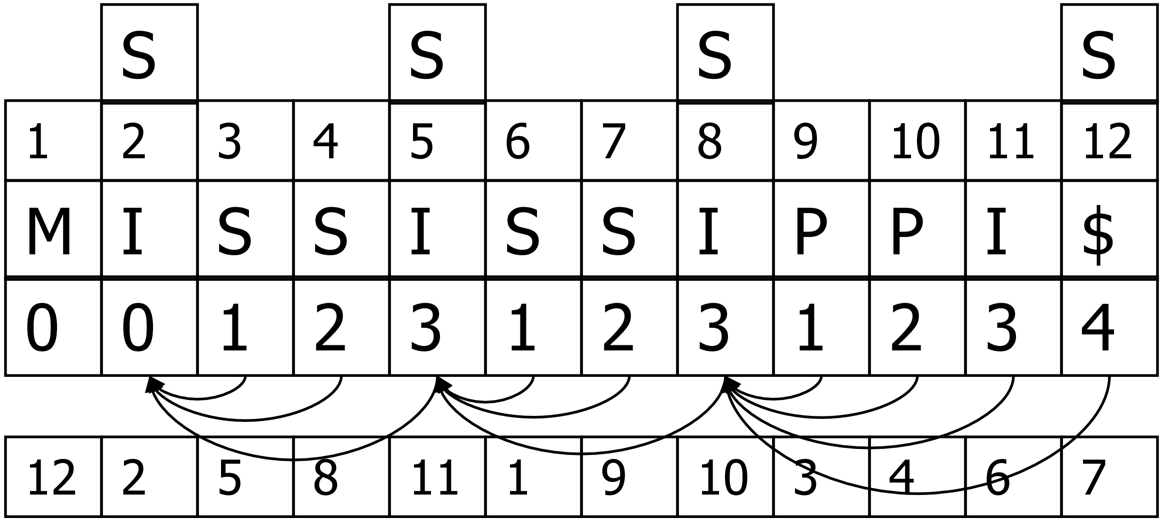
Break into 2 buckets

Break into 2 buckets

Solution

- Observation: Each character participates in the bucket sort at most twice.
  - Type *L* characters only participate in the bucket sort once.
- Solution:
  - Sort all the characters once.
  - Construct m lists according the distance to the closest type *S* character to the left

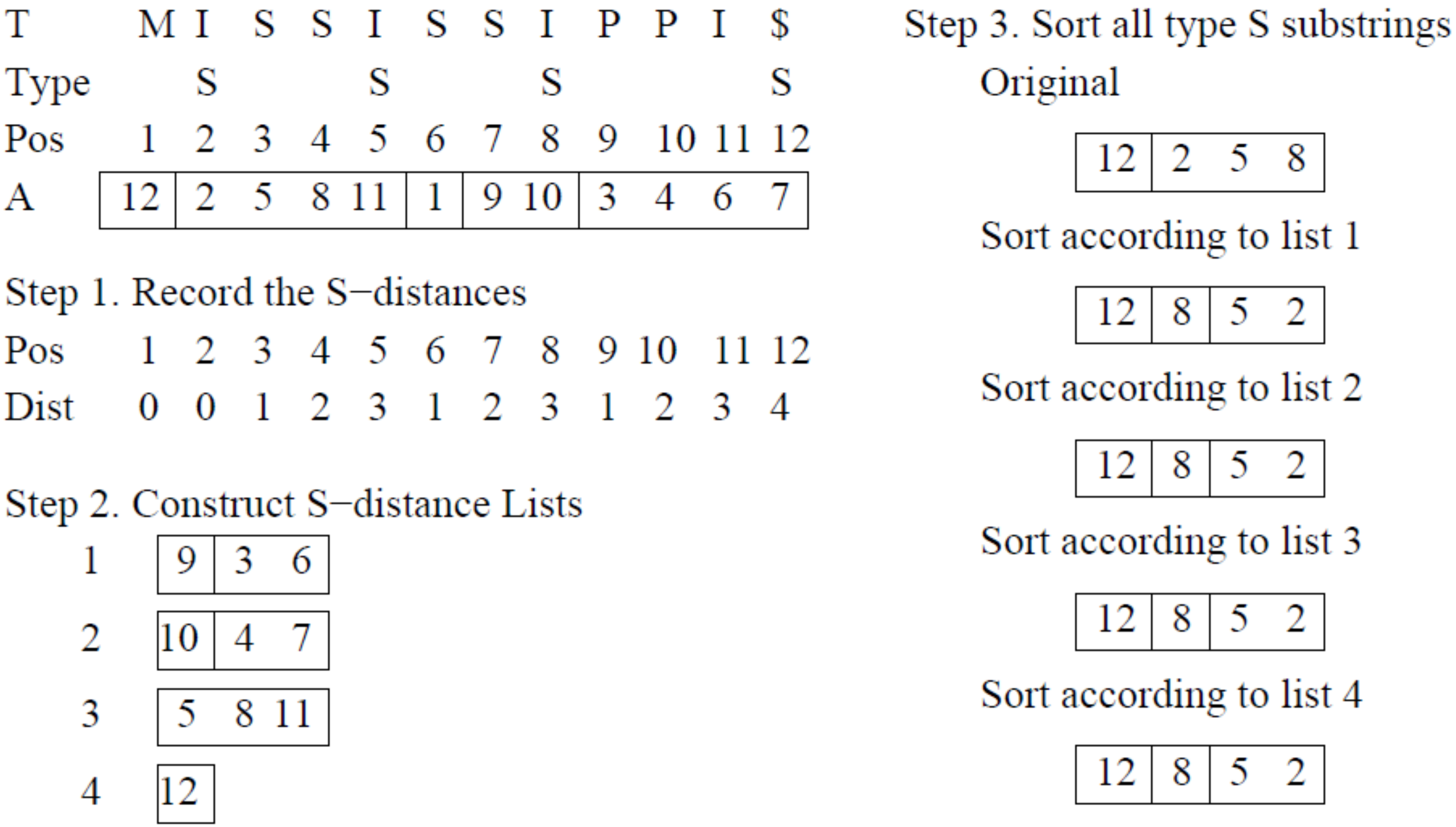
Illustration



Type		S			S			S				S
Index	1	2	3	4	5	6	7	8	9	10	11	12
Text	M	I	S	S	I	S	S	I	P	P	I	$
Distance	0	0	1	2	3	1	2	3	1	2	3	4
Sorted Order of characters	12	2	5	8	11	1	9	10	3	4	6	7

The Lists

Sort the type *S* substrings using the lists



T	M	I	S	S	I	S	S	I	P	P	I	$
Type		S			S			S				S
Pos	1	2	3	4	5	6	7	8	9	10	11	12
A	12	2	5	8	11	1	9	10	3	4	6	7

Step 1. Record the S−distances

Pos	1	2	3	4	5	6	7	8	9	10	11	12
Dist	0	0	1	2	3	1	2	3	1	2	3	4

Step 2. Construct S−distance Lists

1	9	3	6
2	10	4	7
3	5	8	11
4	12		

Step 3. Sort all type S substrings

Original

12	2	5	8

Sort according to list 1

12	8	5	2

Sort according to list 2

12	8	5	2

Sort according to list 3

12	8	5	2

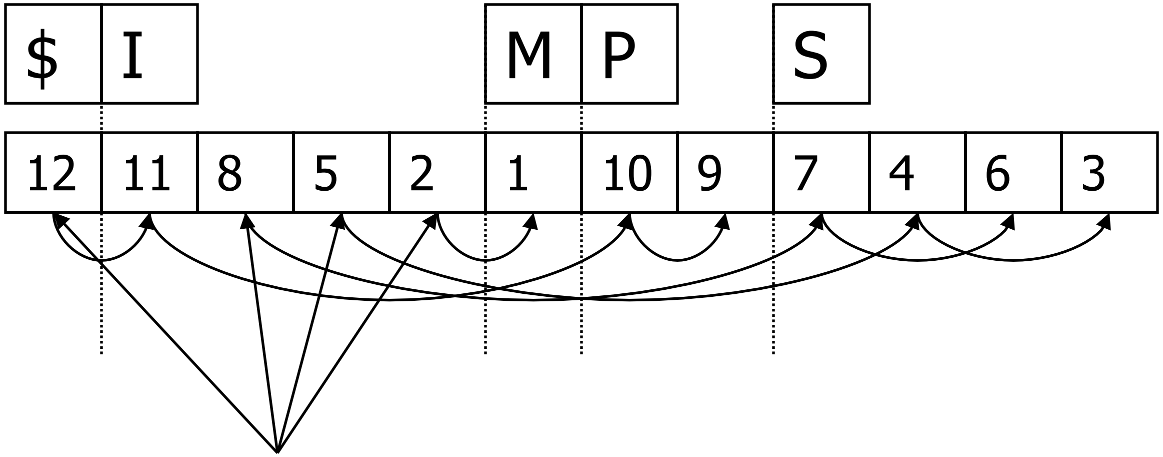
Sort according to list 4

12	8	5	2

Fig. 3. Illustration of the sorting of type *S* substrings of the string MISSISSIPPI$.

Construct Suffix Array for all Suffixes

- The first suffix in the suffix array is a type S suffix.
- For *$1 \leq i \leq n$*, if $T_{SA[i]-1}$ is type *L*, move it to the current front of its bucket
- [$:12][I:2,5,8,11][M:1][P:9,10][S:3,4,6,7]



Sorted order of type *S* suffixes

Run-Time Analysis

- Identify types of suffixes -- O(*n*) time.
- Bucket sort type *S* (or *L*) substrings -- O(*n*) time.
- Construct suffix array from sorted type *S* (or *L*) suffixes -- O(*n*) time.

Exercise

- Consider the popular example string S:
- **`bananainpajamas$`**

1. Construct the suffix array of S using the linear time algorithm
2. Then compute the BWT(S)
3. What's the relationship between the suffix array and BWT ?

Step – Identify the type of each suffix

- LSLSLSSSLSLSLSLL/S
- bananainpajamas$
- 1
- 1234567890123456

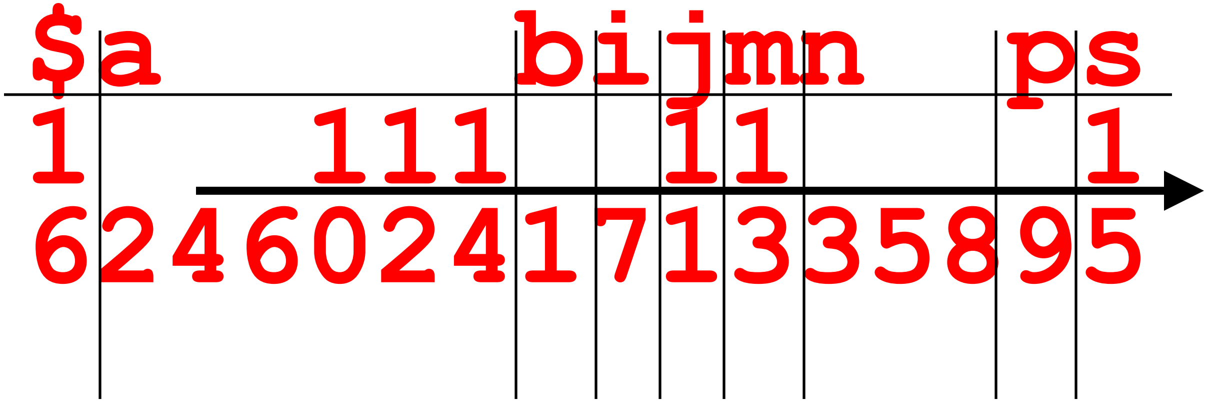
Step – Compute the distance from S

- LSLSLSSSLSLSLSL$_{L/S}$
- bananainpajamas$
- 1111111
- 1234567890123456
- 0012121112121212

Step – Sort order of chars

- LSLSLSSSLSLSLSL$_{L/S}$
- bananainpajamas$
-          1111111
- 1234567890123456
- 0012121112121212
- $a   bijmn ps
- 1   111 11   1
- 6246024171335895

Step – Construct m-Lists



```
■ LSLSLSSSLSLSLSL L/S
■ bananainpajamas$
■          1111111
■ 1234567890123456
■ 0012121112121212
■ $a     bijmn  ps
■ 1   111  11    1
■ 6246024171335895
```

Scan this once and bucket it according to dist.

Step – Generate m-Lists

- List 1
- [7],[11],[13],[3,5,8],[9],[15]
- List 2
- [16],[4,6,10,12,14]

2	0	2	2	2	2	2	2	0	1	1	1	1	1	1	1	1
$	a						b	i	j	m	n			p	s
1				1	1	1			1	1					1
6	2	4	6	0	2	4	1	7	1	3	3	5	8	9	5

Step – Sort S substrings

Bucket the S substrings

[16],[2,4,6,10,12,14],[7],[8]

```
         1111111
1234567890123456
0012121112121212
$a     bijmn  ps
1   111  11    1
6246024171335895
```

Step – Sort S substrings

Bucket the S substrings

[16],[2,4,6,10,12,14],[7],[8]

After using List 1:

[16],[6],[10],[12],[2,4],[14],[7],[8]

List 2 useless. Then?

- **List 1**
- **[7],[11],[13],[3,5,8],[9],[15]**
- **List 2**
- **[16],[4,6,10,12,14]**

Step – Sort S substrings

Bucket the S substrings

[16],[2,4,6,10,12,14],[7],[8]

After using List 1:

[16],[6],[10],[12],[2,4],[14],[7],[8]

List 2 useless. Consider 6 before 4:

[16],[6],[10],[12],[4],[2],[14],[7],[8]

- **List 1**
- **[7],[11],[13],[3,5,8],[9],[15]**
- **List 2**
- **[16],[4,6,10,12,14]**

Step – Generate the Suffix Array

[16],[6],[10],[12],[4],[2],[14],[7],[8]

```
$a     bijmn  ps
1   111  11    1
6246024171335895

$a     ins
1 11  1  1
6602424785
```

Step – Generate the Suffix Array

```
$a     bijmn  ps
1   111  11    1
6246024171335895

$a     in s
1 11  1   1
66024247585
 _      _
```

Step – Generate the Suffix Array

- `$a     bijmn  ps`
- `1   111  11    1`
- `6246024171335895`

- `$a     in ps`
- `1 11  1    1`
- `660242475895`

Step – Generate the Suffix Array

- `$a     bijmn  ps`
- `1   111  11    1`
- `6246024171335895`

- `$a     ijn ps`
- `1 11  1 1   1`
- `6602424715895`

Step – Generate the Suffix Array

```
$a     bijmn  ps
1   111  11    1
6246024171335895

$a     ijn  ps
1 11  1 1    1
66024247153895
```

type S

Step – Generate the Suffix Array

```
$a     bijmn  ps
1   111  11    1
6246024171335895
```

```
$a     bijn  ps
1 11  1  1    1
660242417153895
```

Step – Generate the Suffix Array

```
$a     bijmn  ps
1   111  11    1
6246024171335895
```

```
$a     bijmn  ps
1 11  1  11    1
6602424171353895
```

Final answer

```
bananainpajamas$
         1111111
1234567890123456

Suffix Array:
1 11  1  11    1
6602424171353895
```

Final answer

```
bananainpajamas$
         1111111
1234567890123456

Suffix Array:
1 11  1   11   1
6602424171353895
```

What is the BWT(S) ?

BWT is easy!

```
bananainpajamas$
         1111111
1234567890123456

Suffix Array:
1 11  1  11     1
6602424171353895
BWT:
1  1  11 11     1
5591313660242784
```

BWT construction in linear time

```
bananainpajamas$
         1111111
1234567890123456
```

```
BWT:
1  1  11 11    1
5591313660242784
snpjnbm$aaaaaina
```